6.002 Recitation Notes – Spring 2020

Examples: Large signal and small signal analysis

Kevin Chen, yufengc@mit.edu

Reference: Chapter 7 and 8 of "Foundations of Analog and Digital Electronics Circuits"

Outline:
1. Approach of large signal and small signal analysis
2. Ex: Source follower circuit
3. Ex: Differential amplifier circuit

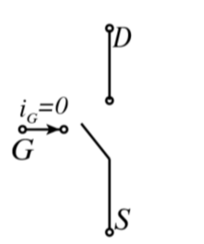
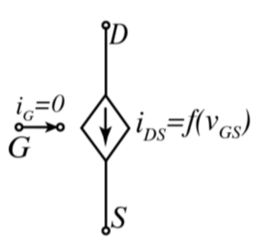
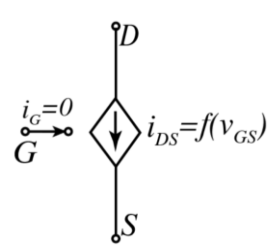
1. Approach of large signal and small signal analysis

We learned about nonlinear elements last week, and this week we will focus on circuits that have MOSFET transistors. MOSFETs are 3-terminal, nonlinear devices. They have several regimes of operation (cut-off, triode, and saturation), and each has a different i-v characteristic. The general approach for solving a MOSFET related problem is summarized as follows:

1. Use large signal analysis to solve for the circuit operating point. This step usually involves solving non-linear equations. The tools are KVL and KCL.
2. Linearize the circuit around the operating point, and use small signal analysis to look at circuit behaviors near the operating point. Since the circuit is linearized, we can use tools such as superposition, Thevenin and Norton equivalence.

1.1 Large signal analysis

A MOSFET has three regimes: cut-off, saturation, and triode. The i-v characteristics and their corresponding conditions are given below:

	Cut-off	Saturation	Triode
Conditions	(1) $v_{GS} < V_T$	(1) $v_{GS} \geq V_T$ (2) $v_{DS} \geq v_{GS} - V_T$	(1) $v_{GS} \geq V_T$ (2) $v_{DS} < v_{GS} - V_T$
i-v characteristic	$i_{DS} = 0$	$i_{DS} = \frac{K}{2}(v_{GS} - V_T)^2$	$i_{DS} = K\left[(v_{GS} - V_T)v_{DS} - \frac{v_{DS}^2}{2}\right]$
Circuit symbol	Open circuit 	Voltage controlled current source 	Voltage controlled current source 

The large signal analysis aims to answer 2 questions:

(1) What is the value of an output variable (most often v_O) at the operating condition?
(2) What are the limits in which this output relationship is true?

For practical purposes, we are mostly interested in operating a MOSFET in its saturation regime.

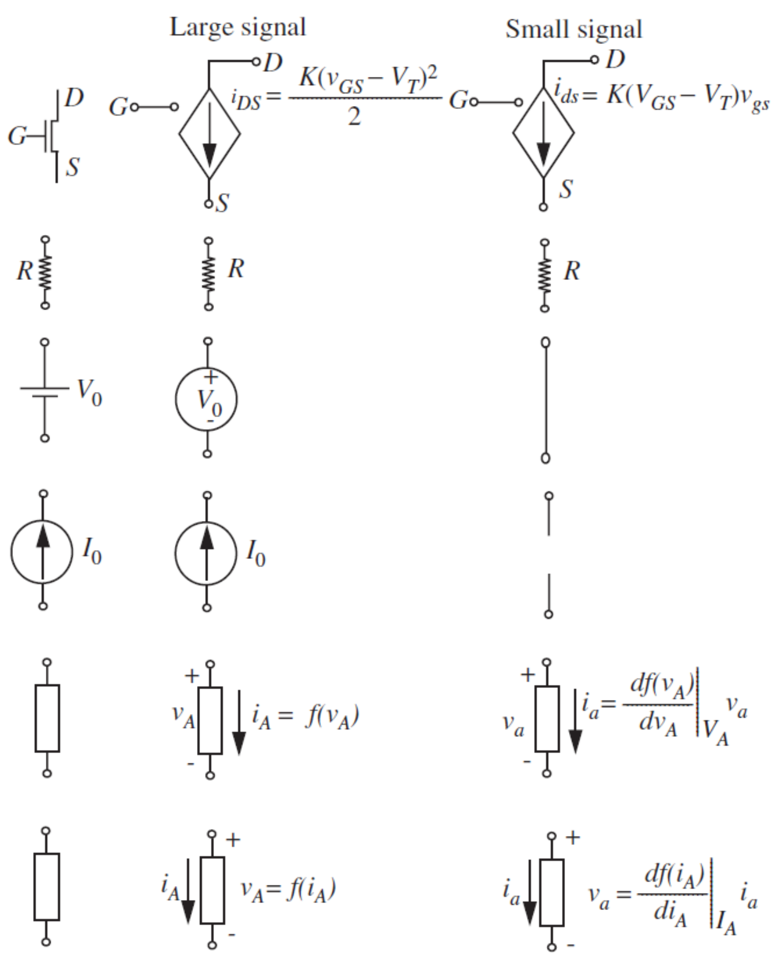
1.2 Small signal analysis

A MOSFET is often used as an amplifier or signal conditioner. It is set to operate in a small interval around an operating point. For this reason, it makes sense to study its behaviors around an operating point. The small signal analysis allows us to linearize a nonlinear element around its OPERATING POINT, and then use all the powerful tools from linear circuit analysis (such as superposition, Thevenin and Norton equivalence).

Performing a small signal analysis requires 3 steps:

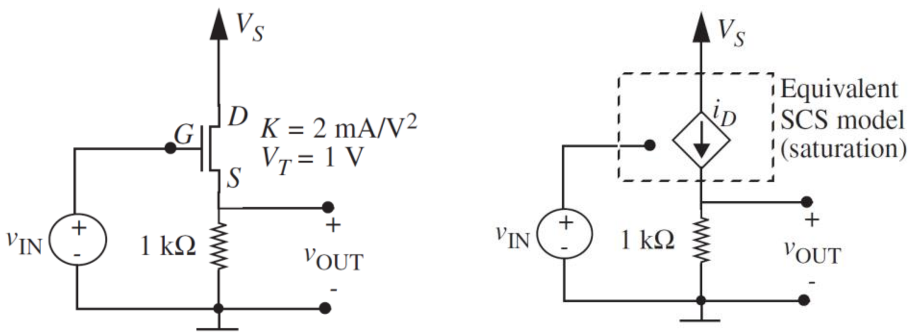
(1) Solve the circuit using large signal analysis at a specific OPERATING POINT
(2) Linearize all nonlinear components at the OPERATING POINT
(3) Simplify the circuit to its small signal equivalent circuit, and solve

The figure below shows how different circuit elements are converted to large signal and small signal models:



2. Example: a source follower circuit

First, let's look at an example that involves 1 MOSFET. The figure below shows the circuit diagram (left) and the large signal model (right).



This circuit is called a MOSFET source follower circuit. The goal is that for a small differential input v_i at the operating condition V_I, the differential output v_o will roughly follow v_i (i.e., $v_o \approx v_i$). We are interested in operating the MOSFET in its saturation regime. Note two things:

(1) This is only true for the differential output v_o around the operating point in the saturation regime
(2) This is better than a "wire" because the input resistance is large and output resistance is small

Step 1: Large signal analysis:
We start by performing large signal analysis by relating v_{OUT} to v_{IN}. We have several equations:

(1) $v_O = i_D R$ (i-v characteristic of the resistor)
(2) $i_D = \frac{K}{2}(v_{GS} - V_T)^2$ (i-v characteristic of the MOSFET in saturation regime)
(3) $v_I = v_{GS} + v_O$ (KVL)

We have three equations and three unknowns (v_O, i_D, v_{GS}), and we can solve these analytically. The nonlinear equation involving v_O is given by:

$$v_O = \frac{K}{2}(v_{IN} - v_O - V_T)^2 R \tag{1}$$

Note this is a quadratic equation of v_O and we can solve it using the quadratic formula.

The second question that large signal analysis aims to answer is in which regimes is equation (1) valid. This will place constraints on our input v_{IN}. In the saturation regime, there are two conditions:

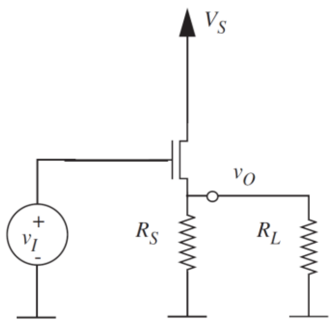
(1) $v_{GS} \geq V_T$ implies $v_{IN} - v_O \geq V_T$, which implies $v_{IN} \geq V_T$ since minimum output is 0.
(2) $v_{DS} \geq v_{GS} - V_T$ implies $V_S - v_O \geq v_{IN} - v_O - V_T$, which implies $V_S + V_T \geq v_{IN}$.

So our analysis is true if our input is in the range:

$$V_T \leq v_{IN} \leq V_S + V_T \tag{2}$$

Equations (1) and (2) completes the large signal analysis. Now let's perform the small signal analysis.

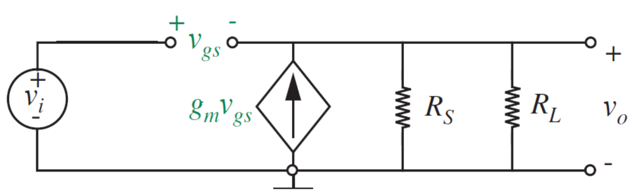
Step 2: Small signal analysis:
In the second step let's linearize the circuit around its operating point. To show illustrate the concept of output resistance, let's add a load resistor to the output of the circuit. The new circuit diagram is as follows:



First, we need to linearize the MOSFET around its operating point. The relationship between i_d and v_{gs} is given by:

$$i_d = \frac{df(v_{GS})}{dv_{GS}}|_{v_{GS}=V_{GS}} v_{gs} = K(V_{GS} - V_T)v_{gs} \quad (3)$$

This converts the MOSFET into a linear voltage controlled current source. Let's define the short-hand notation: $g_m = K(V_{GS} - V_T)$. After we linearize the MOSFET hebavior, let's draw the equivalent small signal circuit based on the figure from page 1. The effective linearized small signal circuit is shown below:



Now let's solve for v_o. We can write two equations:

(1) $v_i = v_{gs} + v_o$ (from KVL)
(2) $g_m v_{gs} = \frac{v_o}{R_S||R_L}$ (from KCL)

There are two unknowns v_o and v_{gs}, and we can solve for v_o in terms of v_i. The solution is:

$$\frac{v_o}{v_i} = \frac{R_L R_S g_m}{R_L + R_S + R_L R_S g_m}$$

Note that if we choose the operating point such as g_m is large (more precisely $g_m R_L R_S \gg R_L$ and $g_m R_L R_S \gg R_S$), then the expression can simplify to:

$$\frac{v_o}{v_i} \approx 1 \tag{4}$$

Equation (4) is an important result. It shows the differential output approximately follows the differential input. So far our circuit behaves as a “wire” because it simply follows the input. Why is this better than a wire?

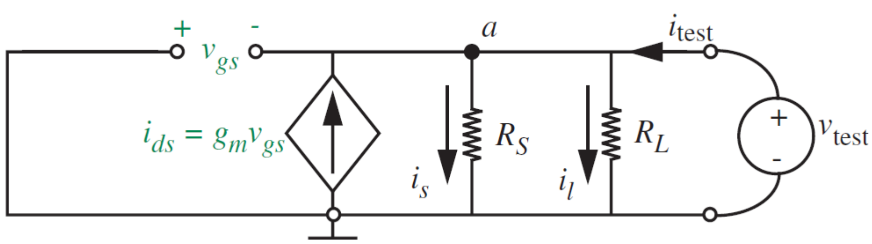
For a realistic circuit, we want to have a large input resistance and a small output resistance so we effectively isolate the influence of the output load on the input source. This is called a buffer –separating different circuit stages and minimizing unwanted influence on each other. Our circuit is a good buffer because it has a large input resistance and a small output resistance. Let’s solve for those components:

Input resistance: r_i

The definition of input resistance is the ratio between a small input voltage v_i and the input current i_i. We are using an ideal MOSFET, and we assume there is no current flow between the gate and source terminals. Hence, $i_i = 0$. Consequently, based on our model, we have $r_i = \infty$.

Output resistance: r_o

Calculating the output resistance r_o involves more steps. We need to put in a test source, and measure the ratio between i_{test} and v_{test}. In this process, we need to turn off other independent sources. This is the same approach as solving for Thevenin equivalence. The equivalent circuit is shown below:



We can write 1 equation based on KCL to relate i_{test} and v_{test}. We have:

$$\text{KCL:} \quad g_m v_{gs} + i_{test} = i_s + i_l$$

We can substitute the values of $v_{gs},\ i_s$ and i_l:

$$g_m v_{test} + i_{test} = \frac{v_{test}}{R_S||R_L}$$

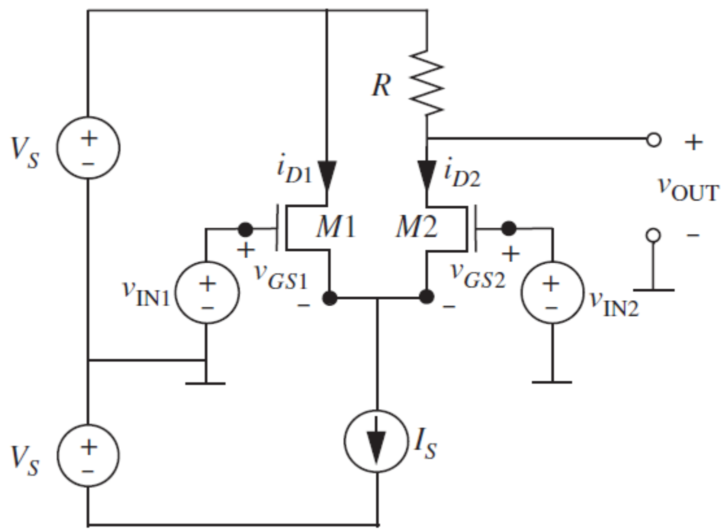
We can simplify the equation to obtain the ratio of i_{test} and v_{test}. We have:

$$r_o = \frac{v_{test}}{i_{test}} = \frac{R_L R_S}{g_m R_L R_S + R_L + R_S} \approx \frac{1}{g_m}$$

If we choose a large g_m (based on the DC operating condition), then we can get a very small output resistance. This shows the reason that a source follower circuit is useful.

3. Example: a differential amplifier circuit

Let's look at a more involved example that has 2 MOSFETs. The circuit below is called a differential amplifier, and it is the building block of an op-amp circuit.



A differential amplifier is used to amplify the difference between 2 input signals v_{IN1} and v_{IN2}. This is an important circuit when we want to read the data coming from a sensor. Sometimes both v_{IN1} and v_{IN2} contain common noise from the surrounding, and the data is the difference between those signals. First, we use large signal analysis to solve for v_{OUT} in terms of the input.

Step 1: Large signal analysis:

In this type of circuit, we assume both MOSFETs are identical and they operate in the saturation regime. We can write down a few equations:

(1) $i_{D1} = \frac{K}{2}(v_{GS1} - V_T)^2$ (i-v characteristic of the MOSFET in saturation regime)

(2) $i_{D2} = \frac{K}{2}(v_{GS2} - V_T)^2$ (i-v characteristic of the MOSFET in saturation regime)

(3) $i_{D1} + i_{D2} = I_S$ (KCL)

(4) $v_{IN1} - v_{GS1} + v_{GS2} - v_{IN2} = 0$ (KVL)

(5) $v_{OUT} = V_S - Ri_{D2}$ (KVL + i-v characteristic of the resistor R)

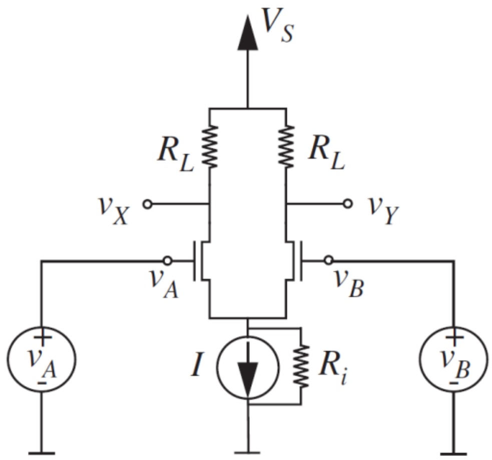
We have 5 equations and 5 unknowns ($i_{D1}, i_{D2}, v_{GS1}, v_{GS2}, and\ v_{OUT}$), and we want to solve for v_{OUT}. We will skip the algebra and simply write the solution of v_{OUT}:

$$v_{OUT} = V_S - \frac{RK}{8}\left(\sqrt{\frac{4I_S}{K} - (v_{IN1} - v_{IN2})^2} - v_{IN1} + v_{IN2}\right)^2 \qquad (5)$$

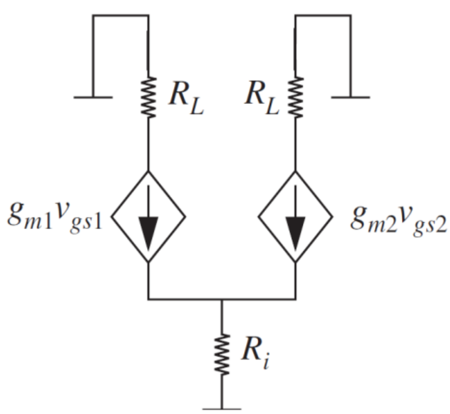
Equation (5) is nonlinear and messy, but there is one important takeaway. The output voltage is a function of the difference of the input: $v_{IN1} - v_{IN2}$. Now let's carry out the small signal analysis.

Step 2: Small signal analysis:

Before solving the small signal analysis, let's add a resistor above MOSFET 1 to make our circuit symmetric. This change is simply there to make the circuit analysis easier. The updated circuit diagram is shown below:



We again linearize the circuit around its operating point. Let's define the conductance: $g_m = K(V_{GS} - V_T)$. In this problem, because of the added resistor R_L, we have symmetry in the large signal solution $V_{GS1} = V_{GS2} = V_{GS}$. The linearized small signal circuit diagram is given below:



Before solving this circuit, we can perform a "basis" transformation. Instead of using the variables v_{in1} and v_{in2}, we can write them as a linear combination of two other variables: their average value v_c and difference v_d. The transformation is given as:

$$\begin{cases} v_c = (v_{in1} + v_{in2})/2 \\ v_d = v_{in1} - v_{in2} \end{cases}$$

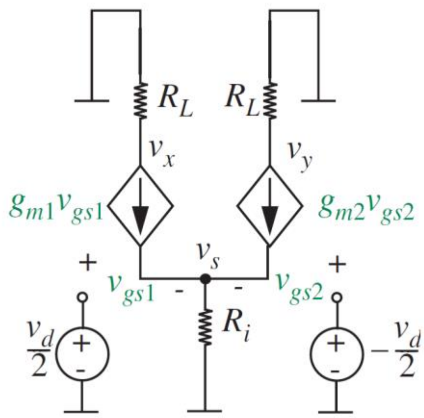
The inverse transformation is:

$$\begin{cases} v_{in1} = v_c + \dfrac{v_d}{2} \\ v_{in2} = v_c - \dfrac{v_d}{2} \end{cases}$$

Now we can use superposition to solve this problem. First, we solve for the case in which one has the input $\frac{v_d}{2}$ and the other has the input $-\frac{v_d}{2}$. Then we solve for the case in which both inputs have the value: v_c. This approach allows us to see how the circuit respond to opposite inputs and common inputs. Finally, we add the two solutions together.

Difference-Mode Model

The small signal model is shown below. Note that here we are interested in solving $v_o = v_x - v_y$.



We can write KVL and KCL equations:

(1) $g_m v_{gs1} + g_m v_{gs2} = \frac{v_s}{R_i}$ (KCL)
(2) $\frac{v_d}{2} - v_{gs1} = v_s$ (KVL)
(3) $-\frac{v_d}{2} - v_{gs1} = v_s$ (KVL)

There are 3 unknowns ($v_s,\ v_{gs1},\ and\ v_{gs2}$) in the equations. Once we solve for these variables, we can solve for vx and vy. We have:

$$\begin{cases} v_x = g_m v_{gs1} R_L = -\dfrac{g_m R_L v_d}{2} \\ v_y = g_m v_{gs2} R_L = \dfrac{g_m R_L v_d}{2} \end{cases}$$

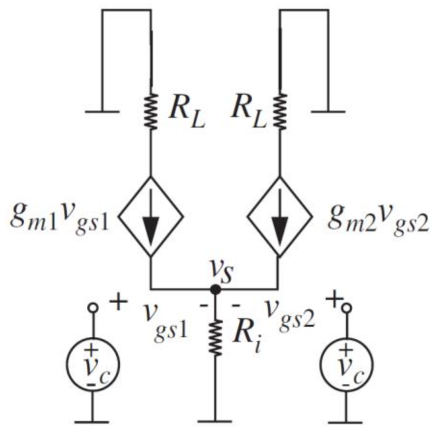
Finally, we can solve for v_o:

$$v_o = v_x - v_y = -g_m R_L v_d \tag{6}$$

Equation (6) is an important result, it shows the output difference signal is amplified by a factor of $-g_m R_L$. If we choose our operating point appropriately, then we can achieve a large amplifying factor.

Common-Mode Model

Next we can solve for the common model. The figure below shows the small signal circuit.



Before we solve for the problem analytically, we can see that based on symmetry, we will have $v_x = v_y$. This implies the output voltage has the value of $v_o = v_x - v_y = 0$.

Now let's confirm this by solving the problems analytically through KVL and KCL.

(1) $g_m v_{gs} + g_m v_{gs} = \frac{v_s}{2R_i}$ (KCL)
(2) $v_c - v_{gs2} = v_s$ (KVL)

We can solve these 2 equations to get:

$$v_{gs} = \frac{1}{2g_m R_i + 1} v_c$$

Finally, we have:

$$v_x = v_y = -v_{gs} g_m R_L \approx -\frac{R_L v_c}{2R_i} \quad (7)$$

Equation (7) is true if $g_m R_i \gg 1$, which again depends on the operating condition. This example shows the differential amplifier amplifies the difference between two signals, and removes the common part. Based on this result, we can further find the effective input and output resistance of this circuit. You can refer to p431-435 of the textbook for more details.