6.002 Recitation Notes – Spring 2020

First Order Filters

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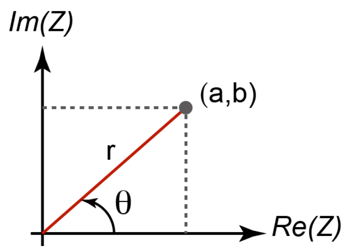
In this recitation we will learn more about first order passive and active filters. We start by a brief review of complex numbers, sinusoidal steady state and impedance method.

Review of Complex Numbers

A complex number, z, can be written as:

$z = a + jb$

where a and b are both real numbers, and j is the imaginary unit defined as $j^2 = -1$. A complex number can be represented as a point in the two-dimensional complex plane.



$a = r\cos\theta$ and $b = r\sin\theta$

$z = a + jb = r\cos\theta + jr\sin\theta = re^{j\theta} = |z|e^{j\angle z}$

(Note that $\cos\theta + j\sin\theta = e^{j\theta}$.)

Magnitude of z: $r = |z| = \sqrt{a^2 + b^2}$

Phase of z: $\theta = \angle z = \tan^{-1}\left(\frac{b}{a}\right)$

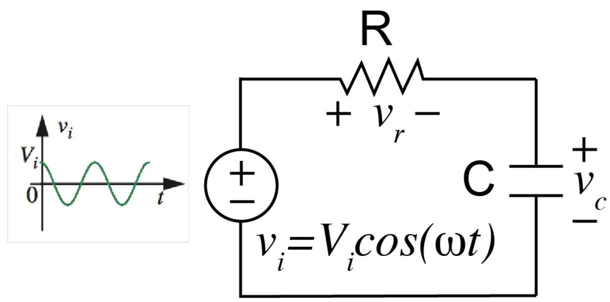
Multiplication and division of complex numbers:

$r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}$

$\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$

Review of Sinusoidal Steady State

We learned about sinusoidal steady state examples previously. As a review lets consider a simple RC circuit shown below.



This is a first order RC circuit shown by the following differential equation:

$$v_i = v_c + RC\frac{dv_c}{dt}$$

As we discussed previously, to solve this differential equation we have to find the homogenous and the particular solutions.

$$v_c = v_{ch} + v_{cp}$$

We will find $v_{ch}$by solving $v_{ch} + RC\frac{dv_{ch}}{dt} = 0$ which yields $v_{ch} = K_1 e^{-t/RC}$ where K_1 can be determined using the initial conditions.

We then will find v_{cp} by solving $v_{cp} + RC\frac{dv_{cp}}{dt} = V_i\cos(\omega t)$ considering that $v_{cp} = K_2\cos(\omega t + \phi)$.

This approach would require us to do quite a lot of trigonometry. Alternatively we can solve this using complex form of v_i and v_c to simplify the problem ($v_i = V_i\cos(\omega t) = Re\{V_i e^{j\omega t}\}$). Note that we often use "s" as a shorthand for "$j\omega$".

We can then find the particular solution by considering $\tilde{v}_i = V_i e^{j\omega t}$ and $\tilde{v}_{cp} = V_{cp}e^{j\omega t}$, and substituting these in the differential equation.
The differential equation simplifies to

$$V_i = V_{cp} + V_{cp}RCj\omega$$

$$V_{cp} = \frac{V_i}{1 + j\omega RC}$$

$$\tilde{v}_{cp} = \frac{V_i}{1+j\omega RC}e^{j\omega t} = \frac{V_i}{\sqrt{1+(\omega RC)^2}}e^{j(\omega t+\phi)} \text{ where } \phi = -\tan^{-1}(\omega RC)$$

Then we can convert from the frequency domain to the time domain if needed.

$$v_{cp}(t) = \frac{V_i}{\sqrt{1+(\omega RC)^2}}\cos(\omega t + \phi)$$

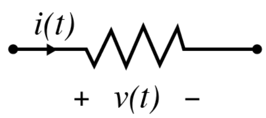
Finally, we find the total solution $v_c = v_{ch} + v_{cp}$. However, for the steady state response we only need to consider v_{cp}.

Analyzing circuits, such as the RC circuit we just reviewed, can even be made simpler if we use the impedance method.

Impedance Method

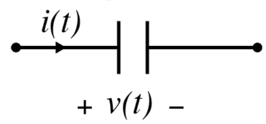
Impedance of an inductor, a capacitor, and a resistor are summarized below. Assume $v = Ve^{st}$ and $i = Ie^{st}$.

Resistor



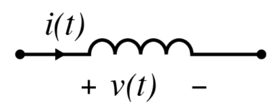
$$v = Ri$$
$$Ve^{st} = RIe^{st}$$
$$V = RI$$
$$Z_R = \frac{V}{I} = R$$

Capacitor



$$i = C\frac{dv}{dt}$$
$$Ie^{st} = C\frac{d}{dt}(Ve^{st})$$
$$Ie^{st} = CVse^{st}$$
$$I = CVs$$
$$Z_C = \frac{V}{I} = \frac{1}{sC} = \frac{1}{j\omega C}$$

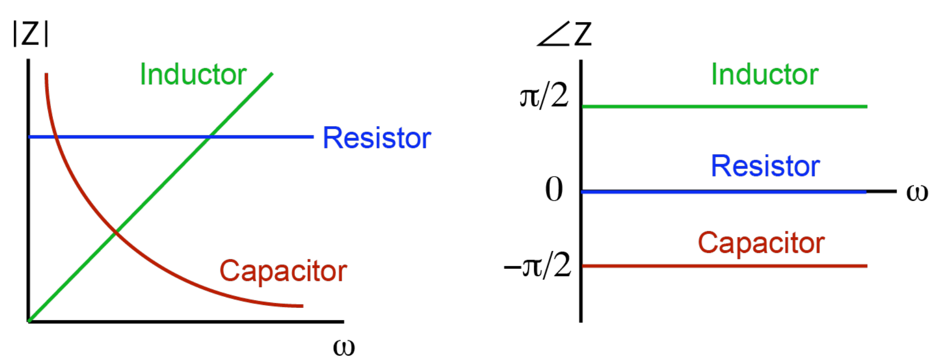
Inductor



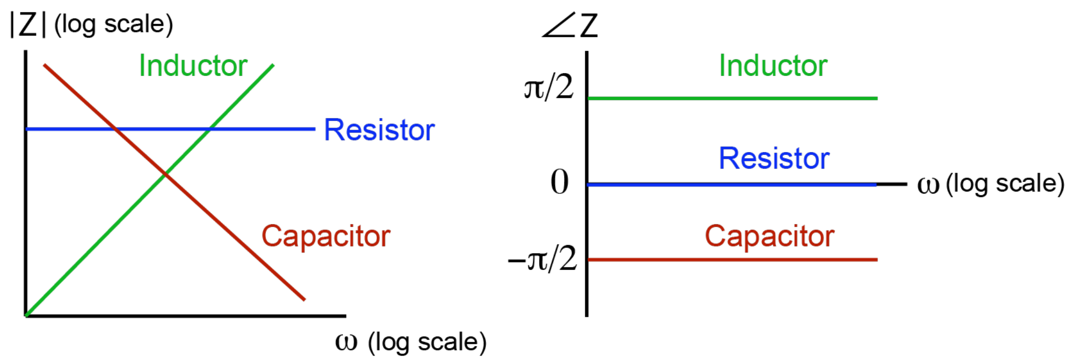
$$v = L\frac{di}{dt}$$
$$Ve^{st} = L\frac{d}{dt}(Ie^{st})$$
$$Ve^{st} = LIse^{st}$$
$$V = LIs$$
$$Z_L = \frac{V}{I} = sL = j\omega L$$

The magnitude and phase of the impedance for these elements are summarized below.

	Impedance	Magnitude	Phase (rad)	Low ω	High ω
Resistor	$Z_R = R$	$\lvert Z_R \rvert = R$	$\angle Z_R = 0$	R	R
Capacitor	$Z_C = \frac{1}{j\omega C}$	$\lvert Z_C \rvert = \frac{1}{\omega C}$	$\angle Z_C = -\frac{\pi}{2}$	Open	Short
Inductor	$Z_L = j\omega L$	$\lvert Z_L \rvert = \omega L$	$\angle Z_L = \frac{\pi}{2}$	Short	Open



This can also be plotted on a logarithmic scale.

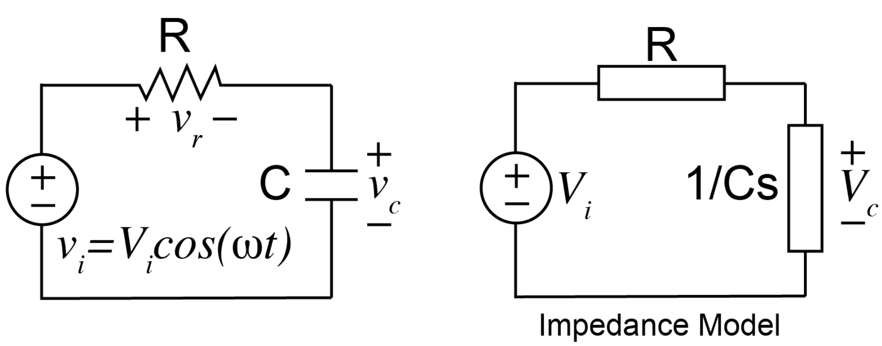


Note: Impedance follows the same combination rules as resistors.

Implementing the impedance method:

1. Replace the sinusoidal sources by their complex (or real) amplitudes. Example: $v_a = V_a \cos(\omega t)$ is replaced by V_a.
2. Replace circuit elements by their impedances.
3. Determine complex amplitude of the voltages and currents by any standard linear circuit analysis technique.
4. If needed, obtain the time variables from the complex amplitudes. For example, $v_O = |V_O| \cos(\omega t + \angle V_O)$.

Now, let's consider the RC circuit again but use the impedance method to analyze it.



$$V_c = \frac{Z_C}{Z_R + Z_C} V_i$$

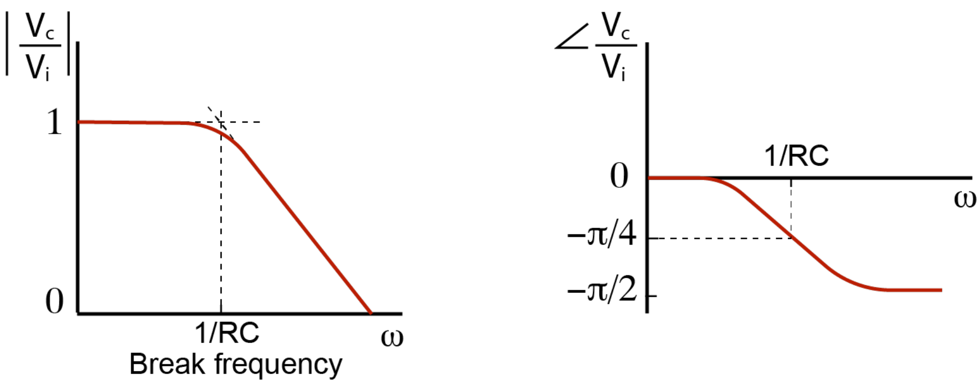
$$V_c = \frac{1/j\omega C}{R + 1/j\omega C} V_i = \frac{1}{j\omega RC + 1} V_i$$

$$H(j\omega) = \frac{V_c}{V_i} = \frac{1}{j\omega RC + 1}$$

$$v_c(t) = |V_c| \cos(\omega t + \angle V_c)$$

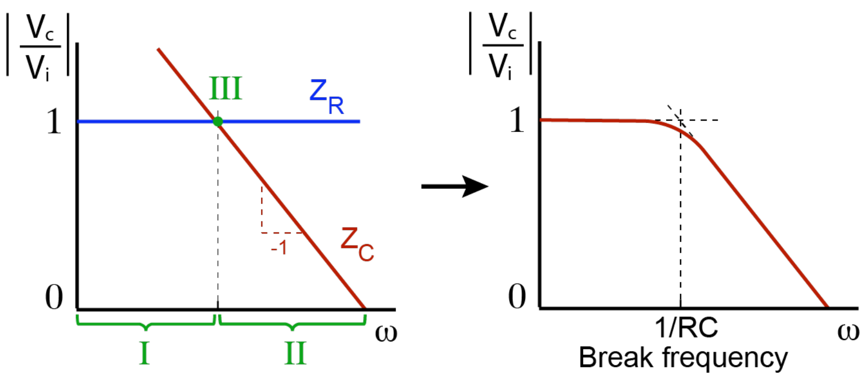
$$v_c(t) = \frac{V_i}{\sqrt{1 + (\omega RC)^2}} \cos(\omega t + \phi) \text{ where } \phi = -\tan^{-1}(\omega RC)$$

The frequency response can then be plotted (magnitude and phase) on a log scale.



Note that the impedance method allows us to intuitively analyze the frequency response of the circuit. To plot the frequency response, we can consider the circuit operation at low and high frequencies and the transition between the two (find the low frequency and high frequency asymptotes). Let's consider the RC example again.

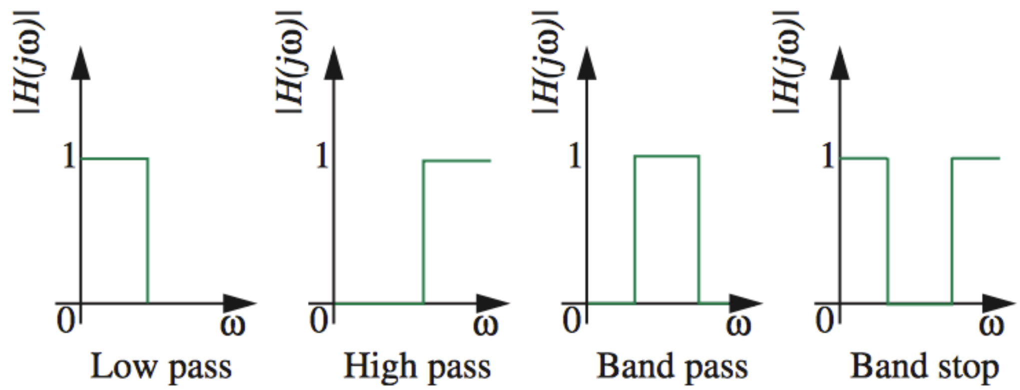
$$H(j\omega) = \frac{V_c}{V_i} = \frac{Z_C}{Z_R + Z_C} = \frac{1}{j\omega RC + 1}$$



Region I	$\frac{V_c}{V_i} = \frac{Z_C}{Z_R + Z_C} \overset{Z_C > Z_R}{\approx} \frac{Z_C}{Z_C} \Rightarrow \left\lvert\frac{V_c}{V_i}\right\rvert \approx 1$
Region II	$\frac{V_c}{V_i} = \frac{Z_C}{Z_R + Z_C} \overset{Z_R > Z_C}{\approx} \frac{Z_C}{Z_R} \Rightarrow \left\lvert\frac{V_c}{V_i}\right\rvert \propto \frac{1}{\omega}$
Region III	$\lvert Z_R\rvert = \lvert Z_C\rvert \Rightarrow R = \frac{1}{\omega C}$ $\left\lvert\frac{V_c}{V_i}\right\rvert = \frac{1}{\sqrt{1 + (\omega RC)^2}} \approx \frac{1}{\sqrt{2}}$

From the frequency response plot, you see that the low frequencies are passed and the high frequencies are attenuated. Thus, the circuit acts as a low pass filter.

Generally, circuits such as this RC circuit can be used as filters to process signals according to their frequency. There are generally 4 types of filters and are shown below.



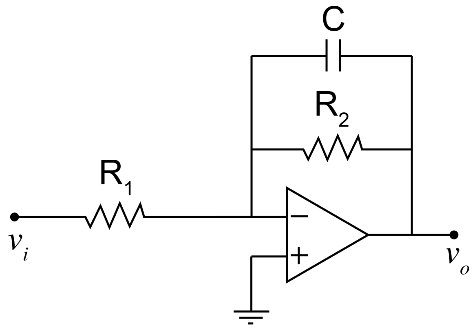
RC Low Pass Filter - Integrator

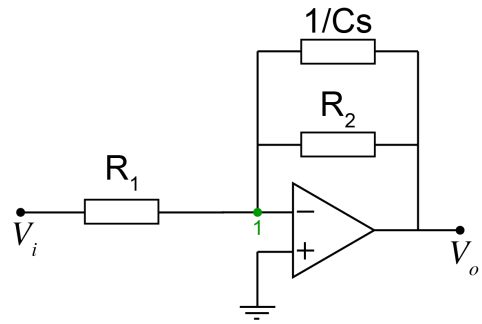
The RC low pass filter can be used as an integrator. Assume an AC input voltage. The output voltage is considered to be the voltage across the capacitor. We consider high frequencies ($\omega \gg \frac{1}{RC}$) so that the capacitor does not have enough time to charge up and the input voltage is approximately equal to the voltage across the resistor ($v_R = v_i$).

$$i_c = C\frac{dv_c}{dt} = i_r = \frac{v_r}{R}$$
$$C\frac{dv_o}{dt} = \frac{v_r}{R} \approx \frac{v_i}{R}$$
$$\frac{dv_o}{dt} \approx \frac{1}{RC}v_i$$
$$v_o \approx \frac{1}{RC}\int v_i\,dt$$

Active Low Pass Filter and Integrator

An active low pass filter is similar in operation to the RC passive filter that we just talked about, except it also has an op-amp in the circuit to help provide amplification and gain control. An example of an active low pass filter is shown below.





Impedance Model

$V_+ = V_- = 0$

Apply KCL at node 1.

$$\frac{V_i}{Z_{R_1}} + \frac{V_0}{Z_{R_2} \parallel Z_C} = 0$$

$$\frac{V_i}{R_1} + V_0(j\omega C + \frac{1}{R_2}) = 0$$

$$\frac{V_i}{R_1} + V_0(\frac{j\omega C R_2 + 1}{R_2}) = 0$$

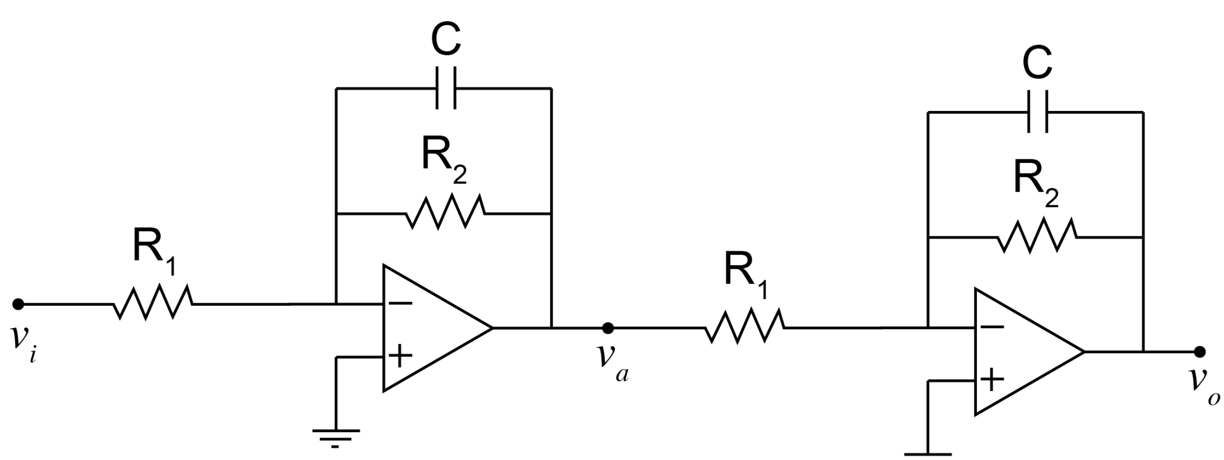
$$V_0 = -\frac{V_i}{R_1}\frac{R_2}{j\omega C R_2 + 1}$$

$$H(j\omega) = \frac{V_0}{V_i} = -\frac{R_2}{R_1}\frac{1}{j\omega R_2 C + 1}$$

Note that this has the same frequency response as the case of the RC passive low pass filter, except with a gain of R_2/R_1.

Cascaded Filters

What will happen if we have two filter cascaded?



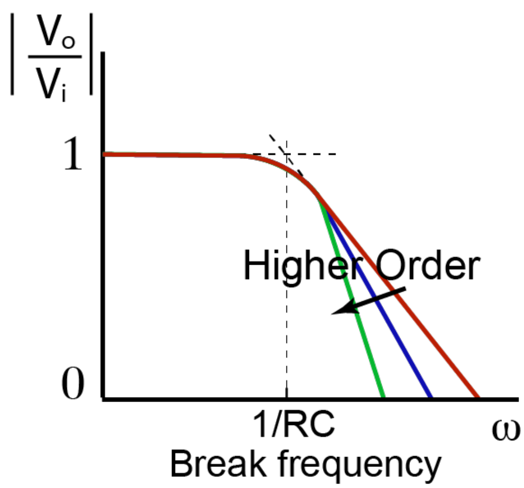
$$H_1(j\omega) = \frac{V_a}{V_i} = -\frac{R_2}{R_1}\frac{1}{j\omega R_2 C + 1}$$
$$V_a = H_1(j\omega)V_i = -\frac{R_2}{R_1}\frac{1}{j\omega R_2 C + 1}V_i$$

$$H_2(j\omega) = \frac{V_o}{V_a} = \frac{V_o}{H_1(j\omega)V_i} = -\frac{R_2}{R_1}\frac{1}{j\omega R_2 C + 1}$$

$$\frac{V_o}{V_i} = H_1(j\omega)H_2(j\omega) = \left(-\frac{R_2}{R_1}\frac{1}{j\omega R_2 C + 1}\right)^2 = \left(\frac{R_2}{R_1}\right)^2 \frac{1}{1 + 2j\omega R_2 C - \omega^2 R_2^2 C^2}$$

The overall transfer function is the product of the transfer functions of the two individual stages. With very high input impedance and very low output impedance, the op-amp can help buffer the input from any effects of the load. This allows it to isolate the different stages of the filter in a cascaded system, as there is no loading of one stage by the next.

In this cascaded filter, for $\omega \gg \frac{1}{R_2 C}$, $\left|\frac{V_o}{V_i}\right| \propto \frac{1}{\omega^2}$.

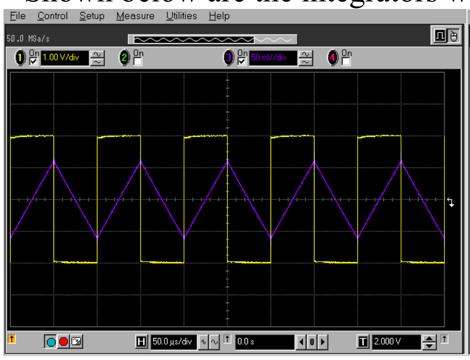


By using a higher order filter, the slope can be made steeper and the filter performance more ideal.

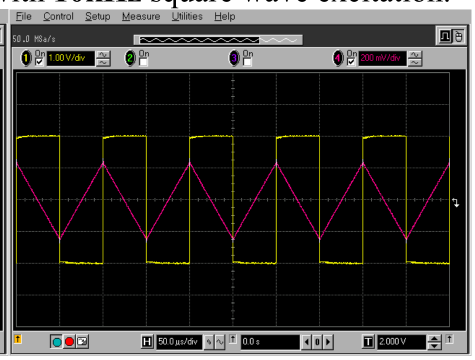
Demo of an Integrator Circuit – Comparing a passive and an active integrator

To test the integrators, a square wave input signal is used. This signal is displayed on channel 1 (yellow) of the scope. The integral of a square wave is a triangle wave. The output of the passive integrator is displayed on channel 3 (purple) and the output of the active integrator is displayed on channel 4 (pink). At high frequencies the input signal is integrated for a short time so the output signal is small. At 10kHz both devices perform well. As the frequency is lowered, the performance of the passive integrator degrades. At 1 kHz noticeable bending in the triangle wave is observed. This is because the voltage across the capacitor has a chance to build up and the input voltage is no longer just the same as the voltage across the resistor. At 100 Hz the passive integrator no longer works correctly because the capacitor is fully charged. The op-amp in the active integrator keeps one end of the input resistor near ground, regardless of the voltage across the capacitor so the integrator continues to properly work.

Shown below are the integrators with **10kHz** square wave excitation.

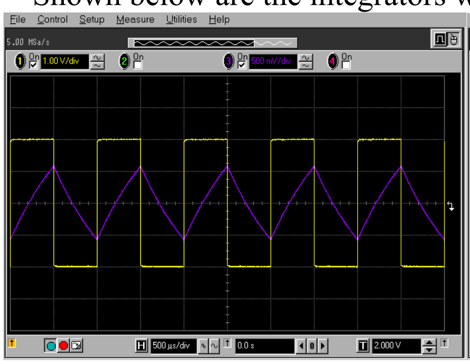


Passive

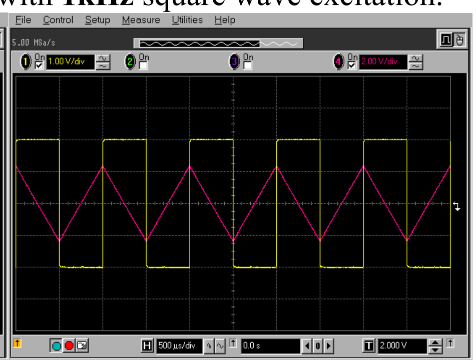


Active

Shown below are the integrators with **1kHz** square wave excitation.

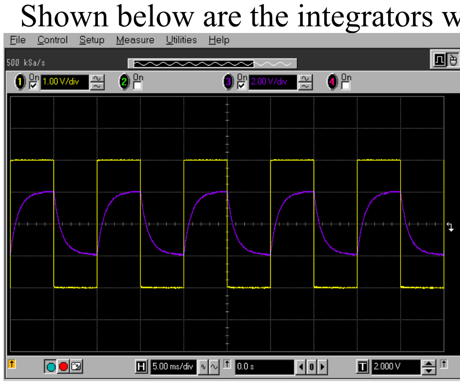


Passive

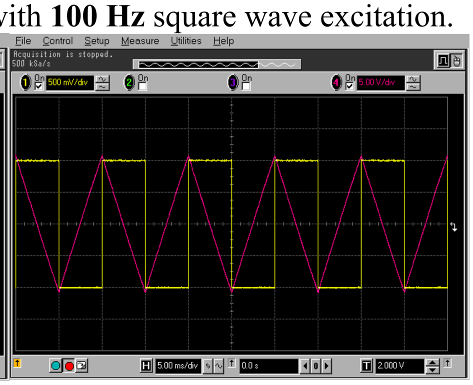


Active

Shown below are the integrators with **100 Hz** square wave excitation.

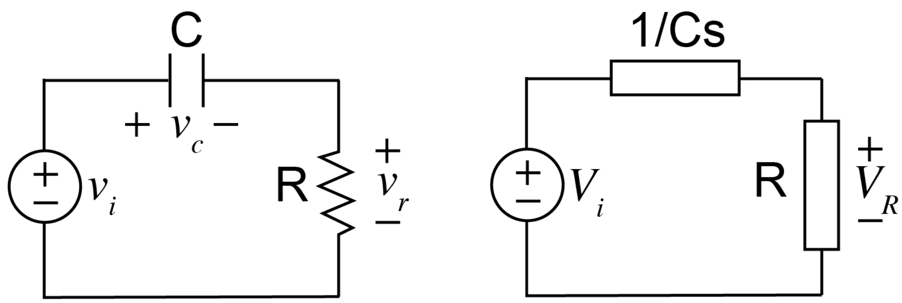


Passive



Active

Passive RC High Pass Filter



Impedance Model

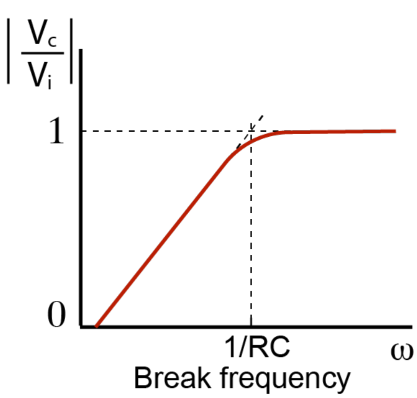
$$V_c = \frac{Z_R}{Z_R + Z_C} V_i$$

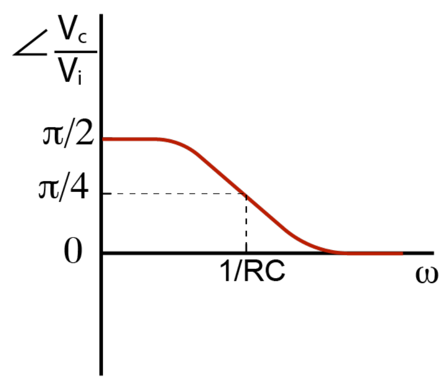
$$V_c = \frac{R}{R + {}^{1}/_{j\omega C}} V_i = \frac{j\omega RC}{j\omega RC + 1} V_i$$

$$H(j\omega) = \frac{V_c}{V_i} = \frac{j\omega RC}{j\omega RC + 1}$$

$$v_c(t) = |V_c| \cos(\omega t + \angle V_c)$$

$$v_c(t) = \frac{V_i \omega RC}{\sqrt{1 + (\omega RC)^2}} \cos(\omega t + \phi) \text{ where } \phi = \tan^{-1}(\frac{1}{\omega RC})$$





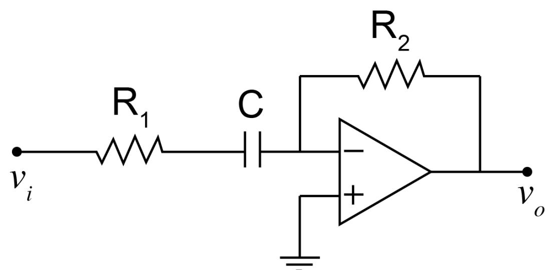
RC High Pass Filter – Differentiator

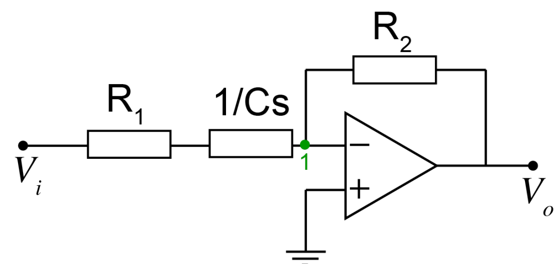
The RC high pass filter can be used as a differentiator. Assume an AC input voltage. The output voltage is considered to be the voltage across the resistor. We consider low frequencies ($\omega \ll \frac{1}{RC}$) so that the capacitor has time to charge up until its voltage almost equals to that of the source ($v_c = v_i$).

$$i_R = i_C = C\frac{dv_c}{dt} = \frac{v_o}{R}$$
$$C\frac{dv_i}{dt} \approx \frac{v_o}{R}$$
$$v_o \approx RC\frac{dv_i}{dt}$$

Active High Pass Filter and Integrator

An active high pass filter is similar in operation to the RC passive filter except it also has an op-amp in the circuit to help provide amplification and gain control. An example of an active high pass filter is shown below





Impedance Model

$V_+ = V_- = 0$

Apply KCL at node 1.

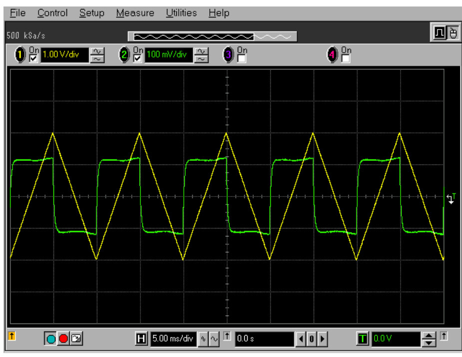
$$\frac{V_i}{Z_{R_1} + Z_C} + \frac{V_0}{Z_{R_2}} = 0$$
$$\frac{V_i}{R_1 + \frac{1}{j\omega C}} + \frac{V_0}{R_2} = 0$$
$$\frac{V_0}{R_2} = -\frac{V_i}{R_1 + \frac{1}{j\omega C}}$$
$$H(j\omega) = \frac{V_0}{V_i} = -\frac{R_2}{R_1 + \frac{1}{j\omega C}} = -\frac{R_2}{R_1}\frac{1}{1 + \frac{1}{j\omega R_1 C}} = -\frac{R_2}{R_1}\frac{j\omega R_1 C}{j\omega R_1 C + 1}$$

Note that this has the same frequency response as the case of the RC passive high pass filter, except with a gain of R_2/R_1.

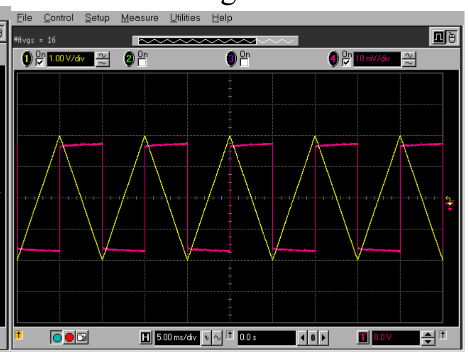
Demo of a differentiator Circuit – Comparing a passive and active differentiator

To test the differentiators, a triangle wave input signal is used. This signal is displayed on the scope in yellow (channel 1). The derivative of a triangle wave is a square wave. The output of the passive differentiator is displayed on channel 2 (green) and the output of the active differentiator is displayed on channel 4 (pink). At low frequencies the rate of rise of the input signal is low so the output signal is small. At 100Hz both devices perform well through the active differentiator has a faster response. As the frequency is raised, the performance of the passive differentiator degrades. At 1kHz it is noticeably slow and at 10kHz, it is unrecognizable. The active differentiator works well at all frequencies.

Shown below are the differentiators with **100Hz** triangle wave excitation.

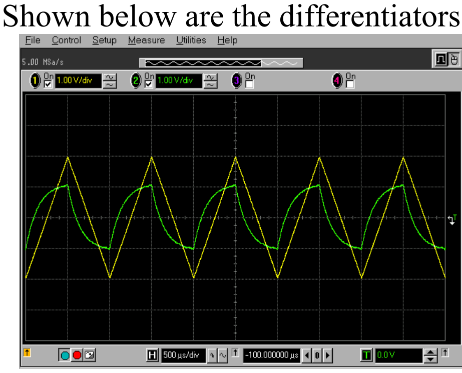


Passive

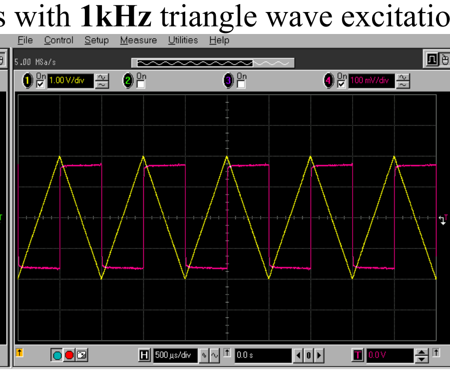


Active

Shown below are the differentiators with **1kHz** triangle wave excitation.

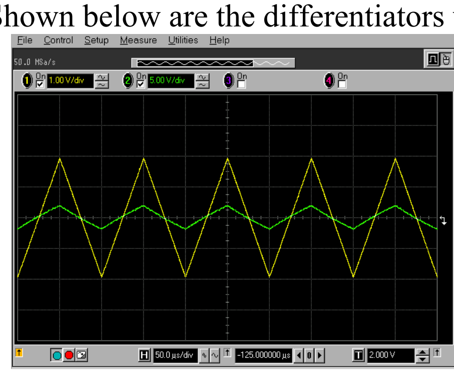


Passive

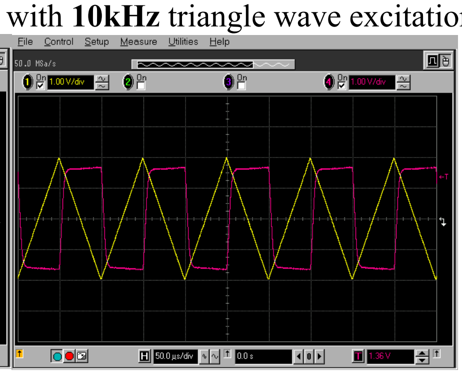


Active

Shown below are the differentiators with **10kHz** triangle wave excitation.



Passive



Active