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6.002 Recitation 2020/04/10

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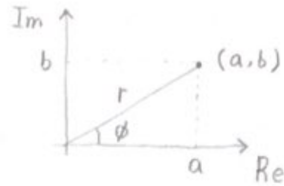
Recitation 8: Impedance, admittance, and graphical representations

1. Review of complex numbers

$Z = a + jb$

complex ↑ real ↑ imaginary ↑



base on the diagram:

$r = \sqrt{a^2+b^2}$ $\quad a = r\cos\phi$

$\phi = \tan^{-1}\left(\frac{b}{a}\right)$ $\quad b = r\sin\phi$

$Z = re^{i\phi}$

magnitude ↑ phase ↑

note: $j^2 = -1$, $\frac{1}{j\omega} = \frac{j}{jj\omega} = -j\frac{1}{\omega}$

2. Impedance

motivation: solving 2nd order differential equations can be tedious. In many senerios, we are interested in the steady-state solution after the transient solution decays. The method of using impedance formulation is very effective for solving the steady-state behaviors of a circuit when the driving source is sinusoidal.

Impedance describes the relationship between the complex voltage ($V_0 e^{st}$) and current ($I_0 e^{st}$) in a circuit element (e.g. resistor, capacitor, inductor)

Defn: $Z = R + jX$

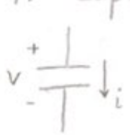
impedance ↑ resistance ↑ reactance ↑

$Y = \frac{1}{Z}$ $\qquad Y = G + jS$

admittance ↑ conductance ↑ susceptance ↑

Let's calculate the impedance of circuit elements

1. Capacitor



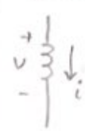
$i = C\frac{dv}{dt}$

$I_0 e^{st} = C\frac{d}{dt}(V_0 e^{st})$

$I_0 e^{st} = Cs V_0 e^{st}$

$\frac{V_0}{I_0} = \frac{1}{Cs}$

2. Inductor



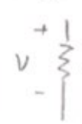
$v = L\frac{di}{dt}$

$V_0 e^{st} = L\frac{d}{dt}(I_0 e^{st})$

$V_0 e^{st} = Ls I_0 e^{st}$

$\frac{V_0}{I_0} = Ls$

3. Resistor



$v = Ri$

$V_0 e^{st} = R I_0 e^{st}$

$\frac{V_0}{I_0} = R$

If we will be using sinusoidal driving sources, let's substitute s with $j\omega$

$\frac{V_0}{I_0} = \frac{1}{j\omega C}$ $\qquad \frac{V_0}{I_0} = j\omega L$ $\qquad \frac{V_0}{I_0} = R$

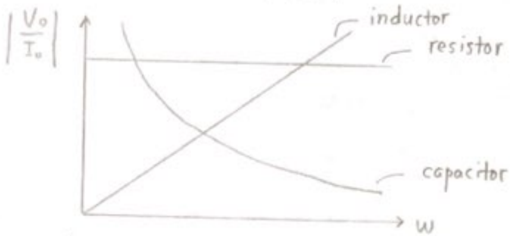
note that $\frac{V_0}{I_0} \equiv Z$ is a complex number (for an resistor, the imaginary part is 0)
We can represent the same information with an amplitude and phase.

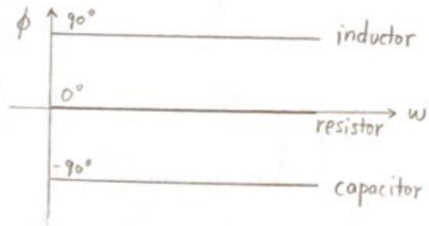
$$\frac{V_0}{I_0} = \left|\frac{V_0}{I_0}\right| e^{j\phi}$$

Capacitor : $\left|\frac{V_0}{I_0}\right| = \frac{1}{\omega C}$ $\quad \phi = -90°$

Inductor : $\left|\frac{V_0}{I_0}\right| = \omega L$ $\quad \phi = 90°$

Resistor : $\left|\frac{V_0}{I_0}\right| = R$ $\quad \phi = 0°$





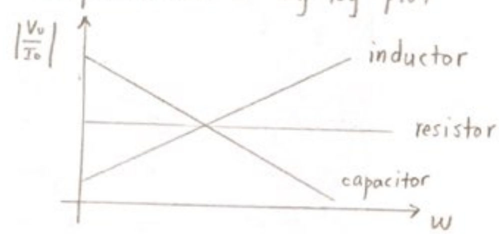
These graphs (amplitude and phase) represent the properties of the circuit elements at different frequencies. What's the behavior at 1Hz? 10Hz? 10000Hz?
It is beneficial to draw the plots on a log-log or semi-log scale.

capacitor : $\log\left|\frac{V_0}{I_0}\right| = \log\left(\frac{1}{\omega C}\right) = -\log C - \log \omega$

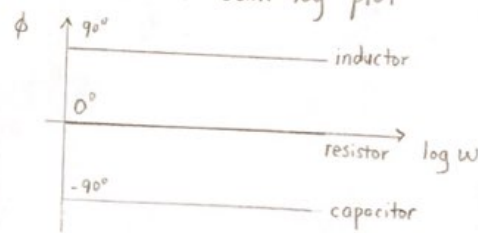
inductor : $\log\left|\frac{V_0}{I_0}\right| = \log \omega L = \log L + \log \omega$

resistor : $\log\left|\frac{V_0}{I_0}\right| = \log R$

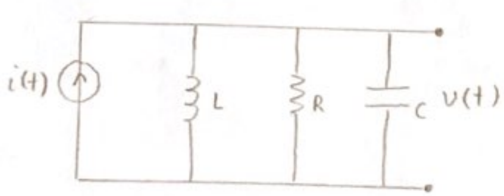
Amplitude on a log-log plot



Phase on a semi-log plot



3. Example : Parallel RLC, sinusoidal response



Given the driving source is $i(t) = I_0 \cos \omega t$, find the steady state solution $v(t)$
note: we know the steady-state solution $v(t) = V_0 \cos(\omega t + \phi)$ has this sinusoidal form, we need to solve for the unknowns V_0, ϕ.

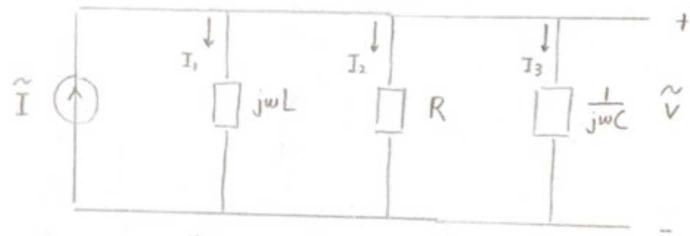
(3) We can solve this equation by directly solving the differential equation, but there is an easier approach.

note: $i(t) = I_o \cos\omega t = Re[I_o e^{j\omega t}]$, let's define $\tilde{I} = I_o e^{j\omega t}$

$v(t) = V_o \cos(\omega t + \phi) = Re[V_o e^{j\omega t + \phi}] = Re[V_o e^{j\phi} e^{j\omega t}] = Re[V_p e^{j\omega t}]$; $\tilde{V} = V_p e^{j\omega t}$

note: $V_p = V_o e^{j\phi}$ ← complex

Now let's re-draw the circuit using impedance representation:



apply KCL: $\tilde{I} = I_1 + I_2 + I_3$

$$\tilde{I} = \frac{\tilde{V}}{j\omega L} + \frac{\tilde{V}}{R} + \frac{\tilde{V}}{1/j\omega C}$$

now substitute $\tilde{I} = I_o e^{j\omega t}$, $\tilde{V} = V_p e^{j\omega t}$

$$I_o e^{j\omega t} = V_p e^{j\omega t}\left[\frac{1}{j\omega L} + \frac{1}{R} + j\omega C\right]$$

$$\frac{V_p}{I_o} = \frac{1}{\frac{1}{j\omega L} + j\omega C + \frac{1}{R}}$$ ← this is a complex number.

$$\frac{V_p}{I_o} = \frac{1}{\frac{1}{R} + j(\omega C - \frac{1}{\omega L})} \cdot \frac{\frac{1}{R} - j(\omega C - \frac{1}{\omega L})}{\frac{1}{R} - j(\omega C - \frac{1}{\omega L})}$$

$$\frac{V_p}{I_o} = \frac{\frac{1}{R} - j(\omega C - \frac{1}{\omega L})}{\frac{1}{R^2} + (\omega C - \frac{1}{\omega L})^2}$$

Let's find the amplitude and phase information:

$$\left|\frac{V_p}{I_o}\right| = \sqrt{\text{real part}^2 + \text{imaginary part}^2} = \frac{1}{\sqrt{(\frac{1}{R})^2 + (\omega C - \frac{1}{\omega L})^2}}$$

$$\phi = \tan^{-1}\left(\frac{\text{imag}}{\text{real}}\right) = \tan^{-1}\left(\frac{-\omega C + \frac{1}{\omega L}}{1/R}\right)$$

Now we can "read off" the steady-state solution:

$$v(t) = \frac{1}{\sqrt{(\frac{1}{R})^2 + (\omega C - \frac{1}{\omega L})^2}} \cos\left(\omega t + \tan^{-1}\left(\frac{-\omega C + \frac{1}{\omega L}}{1/R}\right)\right)$$

Note $v(t)$ changes as ω changes, let's plot amplitude and phase:

when $\omega \approx 0$, $\left|\frac{V_p}{I_o}\right| \approx \frac{1}{\infty} = 0$, $\phi \approx \tan^{-1}(\infty) = 90°$

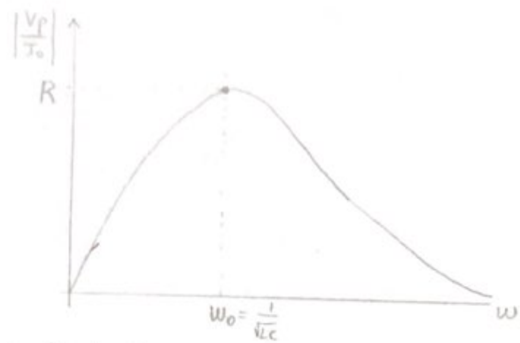
$\omega \approx \infty$, $\left|\frac{V_p}{I_o}\right| \approx \frac{1}{\infty} = 0$, $\phi \approx \tan^{-1}(-\infty) = -90°$

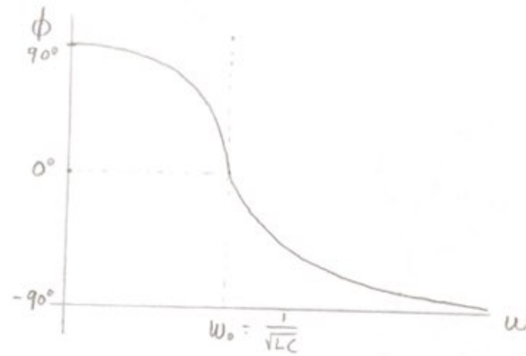
note, when $\omega C = \frac{1}{\omega L}$, $\left|\frac{V_p}{I_o}\right|$ is maximized, $\Rightarrow \omega = \frac{1}{\sqrt{LC}}$ ⇐ natural frequency

when $\omega = \frac{1}{\sqrt{LC}}$, $\phi = \tan^{-1}(0) = 0°$

$\left|\frac{V_p}{I_o}\right| = R$

④ We can plot $\left|\frac{V_p}{I_o}\right|$ vs ω, and ϕ vs ω





Let's find the critical angular frequencies ω_1, and ω_2 such that $\left|\frac{V_p}{I_o}\right| = \frac{1}{\sqrt{2}}\left|\frac{V_p}{I_o}\right|$

solve $\frac{R}{\sqrt{2}} = \frac{1}{\sqrt{(\omega C - \frac{1}{\omega L})^2 + (1/R)^2}}$ $\Rightarrow$ This is equivalent to $(\omega C - \frac{1}{\omega L})^2 = (\frac{1}{R})^2$

we solve for two cases:

① $\omega C - \frac{1}{\omega L} = \frac{1}{R} \Rightarrow \omega^2 - \omega\frac{1}{RC} - \frac{1}{LC} = 0$ (substitute $\alpha = \frac{1}{2RC}$, $\omega_0 = \sqrt{\frac{1}{LC}}$)

$$\omega^2 - 2\alpha\omega - \omega_0^2 = 0$$

$$\omega = \alpha \pm \sqrt{\alpha^2 + \omega_0^2}$$

we take $\omega_1 = \alpha + \sqrt{\alpha^2 + \omega_0^2}$ because we require $\omega > 0$

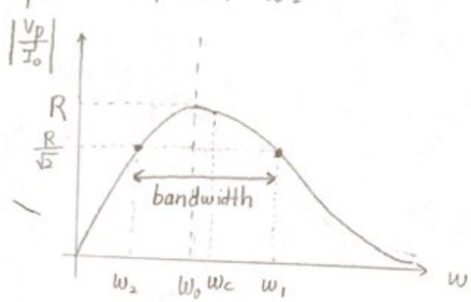
② $\omega C - \frac{1}{\omega L} = -\frac{1}{R} \Rightarrow \omega^2 + \omega\frac{1}{RC} - \frac{1}{LC} = 0$

$$\omega^2 + 2\alpha\omega - \omega_0^2 = 0$$

$$\omega = -\alpha \pm \sqrt{\alpha^2 + \omega_0^2}$$

we take $\omega_2 = -\alpha + \sqrt{\alpha^2 + \omega_0^2}$ because $\omega > 0$

Let's plot ω_1 and ω_2



note $\omega_1 = \alpha + \sqrt{\alpha^2 + \omega_0^2}$

$\omega_2 = -\alpha + \sqrt{\alpha^2 + \omega_0^2}$

defind $\omega_c = \sqrt{\alpha^2 + \omega_0^2}$:= center frequency

bandwidth $= \omega_1 - \omega_2 = 2\alpha$

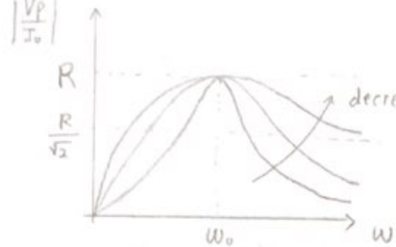
Bandwidth is an important concept. It gives the range of frequencies a circuit is responsive to. The circuit responds strongly to a driving signal whose frequency falls in the range of $[\omega_2, \omega_1]$. For driving signals whose frequency is far from $[\omega_2, \omega_1]$, the response magnitude is very small.

Finally, let's look at the quality factor Q. $Q := \frac{\omega_0}{2\alpha}$

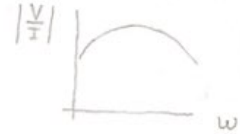
Another interpretation of Q is that $Q = \frac{\text{natural frequency}}{\text{bandwidth}}$

⑤ given ω_0 remains fixed, how does $\left|\frac{V_P}{I_o}\right|$ change when Q changes?

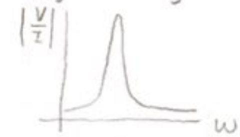
$Q = \frac{\omega_0}{2\alpha}$, as $Q\uparrow$, $\alpha\downarrow$

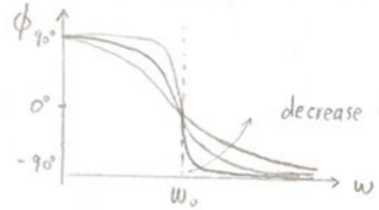


low Q system



high Q system



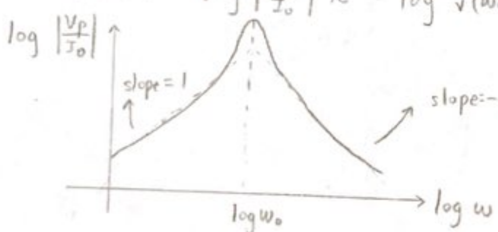


plotting $\left|\frac{V_P}{I_o}\right|$ vs ω requires a computer program. It is convenient to "sketch" the graph by hand. Log-log plots are easier to sketch

Recall $\left|\frac{V_P}{I_o}\right| = \frac{1}{\sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}} \Rightarrow \log\left|\frac{V_P}{I_o}\right| = -\log\sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}$

when $\omega \approx 0$, $\log\left|\frac{V_P}{I_o}\right| \approx -\log\sqrt{\left(\frac{1}{\omega L}\right)^2} = \log\omega + \log L$ (slope of 1)

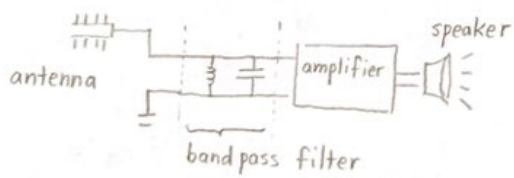
$\omega \approx \infty$, $\log\left|\frac{V_P}{I_o}\right| \approx -\log\sqrt{(\omega C)^2} = -\log\omega + -\log C$ (slope of 1)

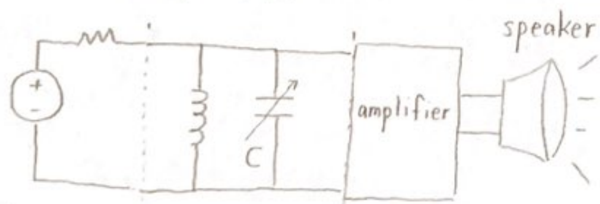


* The rate of increase & decay (slopes) is very important for circuit designs

What is this circuit (parallel RLC) good for?

e.g. Amplitude Modulation (AM) Radio Receiver





Antenna + driving source

goal: use the tunable capacitor to select the frequency of interest