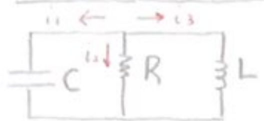
6.002 Recitation 04/03/2020

Series and parallel RLC circuit

Derivation of circuit equations

parallel RLC circuit



note: $V_C = V_R = V_L$

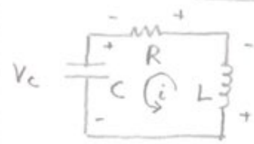
apply KCL: $i_1 + i_2 + i_3 = 0$

$$C\frac{dV}{dt} + \frac{V}{R} + \frac{1}{L}\int_{-\infty}^{t} V(\tau)\,d\tau = 0$$

$$C\frac{d^2V}{dt^2} + \frac{1}{R}\frac{dV}{dt} + \frac{1}{L}V = 0$$

$$\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{1}{LC}V = 0$$

series RLC circuit



apply KVL: $V_C + V_R + V_L = 0$

$$V_C + iR + L\frac{di}{dt} = 0 \qquad \text{note } i = C\frac{dV_C}{dt}$$

$$V_C + RC\frac{dV_C}{dt} + LC\frac{d^2V_C}{dt} = 0$$

$$\frac{d^2V_C}{dt^2} + \frac{R}{L}\frac{dV_C}{dt} + \frac{1}{LC}V_C = 0$$

note: there is a difference between the coefficients in front of the $\frac{dV}{dt}$ term.

To solve the differential equations, we want to rewrite them in the form:

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0 \qquad \text{or} \qquad \ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x = 0$$

where we define:

unit has the dimension of

- α: damping coefficient, [=] 1/s
- ω_0: natural frequency, [=] 1/s
- Q: quality factor, [=] → dimensionless

parallel RLC circuit

$$\frac{1}{RC} = 2\alpha \Rightarrow \alpha = \frac{1}{2RC}$$

$$\omega_0^2 = \frac{1}{LC} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0}{2\alpha} = R\sqrt{\frac{C}{L}}$$

series RLC circuit

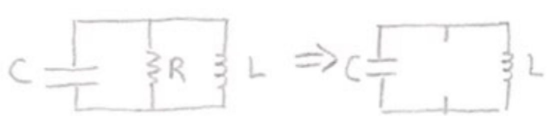
$$\frac{R}{L} = 2\alpha \Rightarrow \alpha = \frac{R}{2L}$$

$$\omega_0^2 = \frac{1}{LC} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0}{2\alpha} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Note the difference in α, and Q. We can check if it makes sense by taking the limit and try to turn it into a LC circuit

parallel RLC circuit



we need to increase $R \to \infty$

In the LC circuit limit:

$$\lim_{R\to\infty} \alpha = 0$$

$$\lim_{R\to\infty} Q \to \infty$$

series RLC circuit

we need to reduce $R \to 0$

$$\lim_{R\to 0} \alpha = 0$$

$$\lim_{R\to 0} Q \to \infty$$

Now let's solve the equation:

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0 \qquad (1)$$

This is an undriven case ($RHS = 0$). We only need to solve for the homogeneous solution. Let's guess that the solution has the form:

$$A e^{st} \qquad (2)$$

Then we can substitute this back into equation (1)

$$A \underbrace{(s^2 + 2\alpha s + \omega_0^2)}_{\text{characteristic equation}} e^{st} = 0$$

We can solve this quadratic equation by using the formula:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can solve for the two roots of the characteristic eqn

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Plug this back into eqn (2), we have

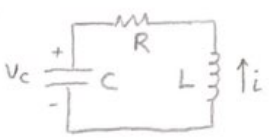
$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$= A_1 e^{-\alpha + \sqrt{\alpha^2 - \omega_0^2}\, t} + A_2 e^{-\alpha - \sqrt{\alpha^2 - \omega_0^2}\, t}$$

where A_1, A_2 are the coefficients. Their values can be determined by using the initial conditions.

(Reminder: A 2^{nd} order ODE needs two initial conditions)

Ex: Let's consider a series RLC circuit, with the initial conditions $v_c(t=0)$, $i(t=0)$



d.e.g. $\frac{d^2 V_c}{dt} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC} V_c = 0$

$$\ddot{V_c} + 2\alpha \dot{V_c} + \omega_0^2 V_c = 0$$

substitute:

$$V_c(t=0) = V_c(0) = A_1 + A_2$$

$$\dot{V_c}(t=0) = \frac{1}{C} i(0) = s_1 A_1 + s_2 A_2 \qquad \left(\text{note: } i = C\frac{dv}{dt} \Rightarrow \frac{dv}{dt}(t=0) = \frac{1}{c} i(0)\right)$$

solve for A_1 and A_2:

$$A_1 = \frac{C s_2 V_c(0) - i(0)}{C(s_2 - s_1)}$$

$$A_2 = \frac{C s_1 V_c(0) - i(0)}{C(s_1 - s_2)}$$

The solution is: $V_c(t) = \frac{C s_2 V_c(0) - i(0)}{C(s_2 - s_1)} e^{s_1 t} + \frac{C s_1 V_c(0) - i(0)}{C(s_1 - s_2)} e^{s_2 t}$

③ Let's look at large time behavior:

Recall $s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$. This means $Re(s) < 0$, so the solution decays exponentially to 0.

$\lim_{t \to \infty} V_c(t) = 0$

This means all energy will be slowly dissipated in a RLC circuit.

Let's now look at small time behavior and energy dissipation.

↳ Based on the root equation, $s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$, let's define three cases:

1. $\alpha < \omega_0 \Rightarrow$ under-damped
2. $\alpha > \omega_0 \Rightarrow$ over-damped
3. $\alpha = \omega_0 \Rightarrow$ critically-damped

1. Under-damped

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}, \quad \alpha < \omega_0 \Rightarrow \sqrt{\alpha^2 - \omega_0^2} < 0$$

$$= -\alpha \pm j\omega_d \qquad \text{where we define } \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Let's look at our general solution:

$$x(t) = A_1 e^{-\alpha + \sqrt{\alpha^2 - \omega_0^2}\, t} + A_2 e^{-\alpha - \sqrt{\alpha^2 - \omega_0^2}\, t}$$

$$= A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

note: $e^{j\omega_d t} = \cos \omega_d t + i \sin \omega_d t$

$$x(t) = A_1 e^{-\alpha t}\left[\cos \omega_d t + j \sin \omega_d t\right] + A_2 e^{-\alpha t}\left[\cos \omega_d t - j \sin \omega_d t\right]$$

$$= B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

where B_1 and B_2 are two coefficients we need to solve based on initial conditions.

2. Over-damped:

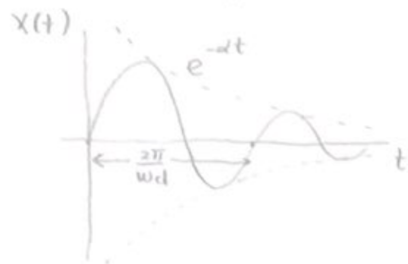
$$x(t) = A_1 e^{-\alpha + \sqrt{\alpha^2 - \omega_0^2}\, t} + A_2 e^{-\alpha - \sqrt{\alpha^2 - \omega_0^2}\, t}$$

3. Critically-damped:

If we have repeated roots ($\alpha = \omega_0 \Rightarrow s_1 = s_2$), then we need to look for another solution. We skip the derivation details here, but you can check that $e^{-\alpha t}$, and $te^{-\alpha t}$ are solutions.

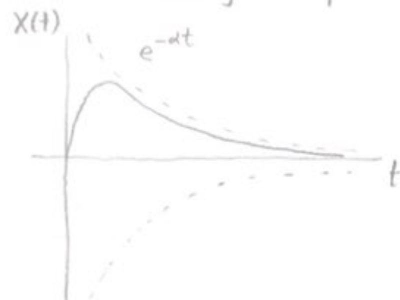
$$x(t) = A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}$$

④ Under-damped



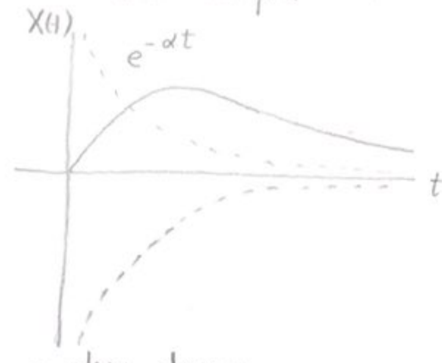
- fast decay
- with oscillation

Critically-damped



- fast decay
- no oscillation
- desireable for electrical & mechanical systems

Over-damped



- slow decay
- no oscillation

Let's return to discuss about the physical interpretation of the quality factor Q

$$Q \equiv \frac{\omega_0}{2\alpha}$$

note: these interpretations only make sense for underdamped cases:

① $x(t) \sim X_0 e^{-\alpha t}$, let's choose $t =$ Q periods of oscillation $= \frac{Q}{f} = \frac{2\pi Q}{\omega_d}$

$$x(t = \tfrac{2\pi}{\omega_d}Q) \sim X_0\, e^{-\alpha \frac{2\pi}{\omega_d} Q} = X_0\, e^{-\alpha \frac{2\pi}{\omega_d} \frac{\omega_0}{2\alpha}} = X_0\, e^{-\pi \frac{\omega_0}{\omega_d}}$$

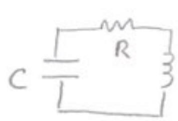
often time we have $\omega_0 >> \alpha$, which implies $\omega_0 \approx \omega_d$

$$X(t) \sim X_0\, e^{-\pi} \approx 4\%$$

The amplitude of state variables (e.g. $v(t)$, $i(t)$) decay to approximately 4% of their original values after Q cycles of oscillations

② energy interpretation:

In a series RLC circuit, the energy is given by:



$$W_T(t) = W_E(t) + W_M(t) = \tfrac{1}{2} C v_c^2(t) + \tfrac{1}{2} L i_L^2(t)$$

electrical energy in the capacitor; magnetic energy in the inductor

$$\approx \left(\tfrac{1}{2} C v_c^2(0) + \tfrac{1}{2} L i^2(0)\right) e^{-2\alpha t}$$

energy at time T_0: $E_0 e^{-2\alpha T_0}$

energy at time $T_0 + \frac{2\pi}{\omega_d}$: $E_0 e^{-2\alpha (T_0 + \frac{2\pi}{\omega_d})}$

Let's look at a ratio:

$$\frac{\text{energy at } T_0}{\text{energy lost from } \underbrace{T_0 \text{ to } T_0 + \tfrac{2\pi}{\omega_d}}_{\text{one period}}} = \frac{E_0 e^{-2\alpha T_0}}{E_0 e^{-2\alpha T_0} - E_0 e^{-2\alpha (T_0 + \frac{2\pi}{\omega_d})}} = \frac{1}{1 - e^{-2\alpha \frac{2\pi}{\omega_d}}} \overset{\omega_0 \approx \omega_d}{\approx} \frac{1}{1 - e^{-\frac{2\pi}{Q}}}$$

let's use the formula: $\frac{1}{1-e^{-x}} = \frac{1}{1 - [1 - x + \frac{x^2}{2!} - \cdots]} \approx \frac{1}{x}$ for small x

$\frac{1}{1 - e^{-2\pi/Q}} \approx \frac{Q}{2\pi}$; This implies $Q \approx 2\pi \frac{\text{energy stored}}{\text{energy lost in 1 period}}$

Take away: ① Q tells you about system efficiency ② You can estimate Q graphically

Supplementary information for Recitation on RLC circuits

Here we show simulation results of a series RLC circuit. The values of the resistor, capacitor, and inductors are:

R = 20kΩ, L = 1μH, and C = 10μF.

As discussed in the recitation, the differential equation that describes this system is:

$$\frac{d^2v}{dt}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=0.$$

The values of the damping coefficient, natural frequency, driving frequency, and quality factors are:

$$\alpha = 1e10\ s^{-1}, w_0 = 1e11\ s^{-1},\ w_d = 9.95e10\ s^{-1},\ Q = 5,$$

Note that we have the condition $w_0 \approx w_d$. Figure 1a shows the voltage v(t) and current i(t) in the circuit.

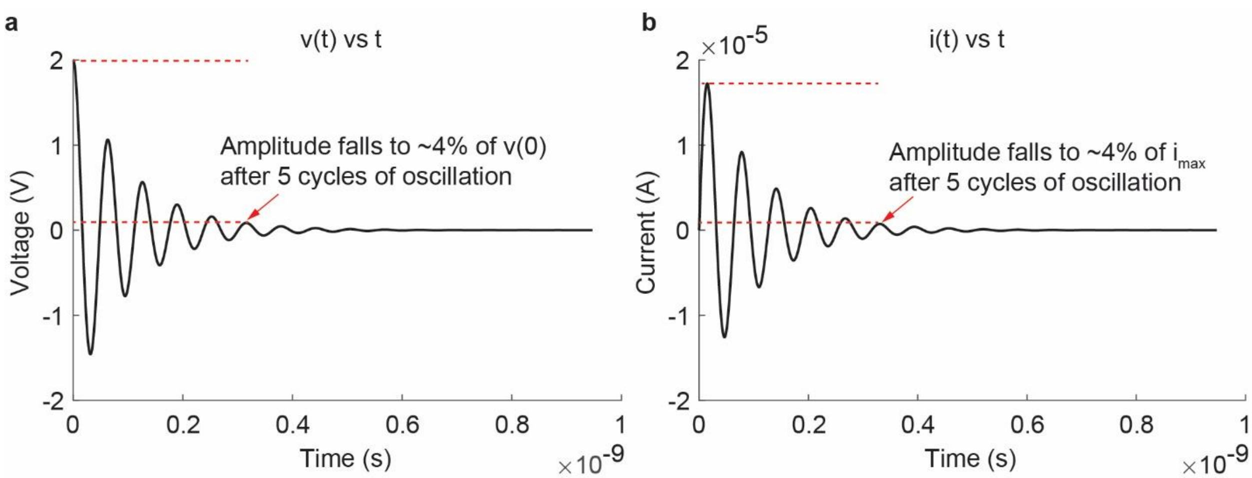


Figure 1. Simulated v(t) and i(t) for a series RLC circuit with R = 20kΩ, L = 1μH, and C = 10μF.

The Q factor of this circuit is 5, so the amplitudes decay to about 4% of the original values in 5 periods of oscillation. In addition, the fraction of energy dissipated is 72% per cycle.

We can change the values of a few circuit elements and observe changes in the voltage and current profiles. For instance, we can reduce R by a factor of 2. Changing R from 20kΩ to 10kΩ reduces the damping coefficient and increases quality factor by a factor of 2. Figure 2 a and b show the voltage and current profiles of the new circuit. Note that now it takes approximately 10 cycles (because Q=10) for the voltage and current amplitudes to reduce to 4% values. In addition, the fraction of energy dissipated is now 47% per cycle. This shows a high Q system is less lossy. From an efficiency perspective, it is favorable to build electrical or mechanical system of high Q values.

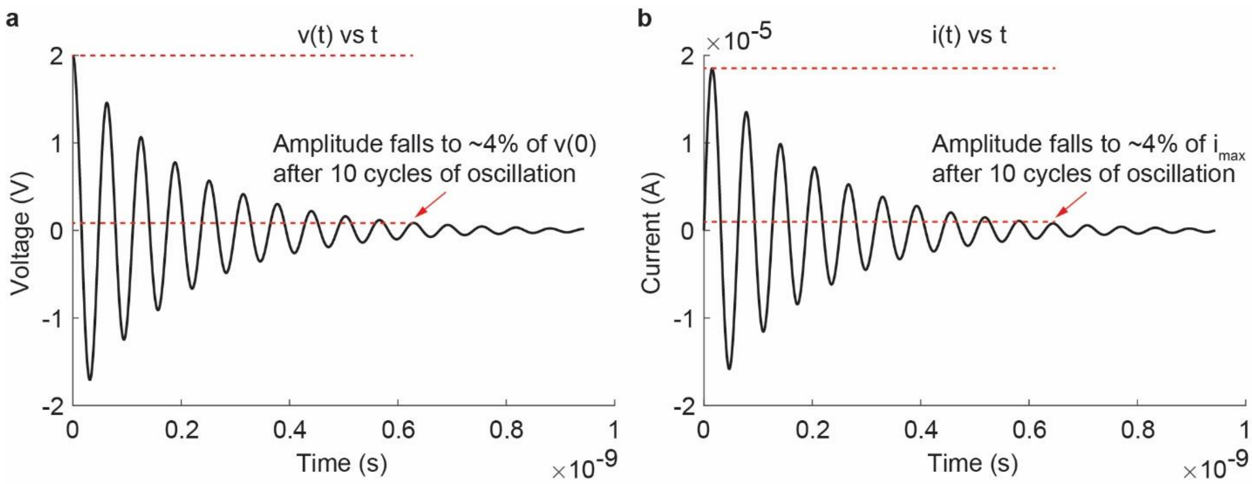


Figure 2. Simulated v(t) and i(t) for a series RLC circuit with R = 10kΩ, L = 1μH, and C = 10μF.