6.002 Recitation Notes – Spring 2020

Inductors and RL Networks

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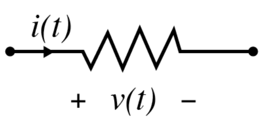
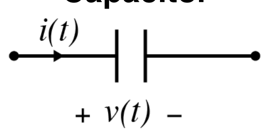
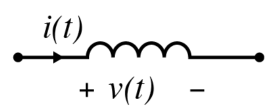
Reference: "Foundations of Analog and Digital Electronics Circuits", Chapters 9, Chapter 10

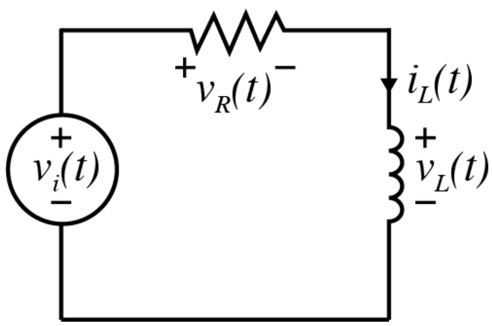
Energy Storage Elements – Capacitors and Inductors

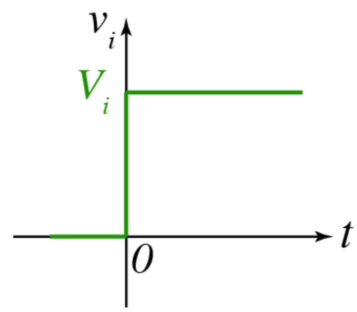
The three main circuit elements that we work with in 6.002 are resistors, capacitors and inductors. Unlike resistors, capacitors and inductors store energy rather than dissipate it. The device laws for these elements are summarized below.
More specifically, an inductor is a passive two-terminal element, consisting of a wire wound into a coil around a core, that stores energy in a magnetic field when electric current flows through it. An inductor opposes sudden changes in current.

Resistor	**Capacitor**	**Inductor**
		
$v(t) = Ri(t)$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = L\frac{di(t)}{dt}$
R is the resistance, with units of Ohm (Ω).	C is the capacitance, with units of Farads (F).	L is the inductance, with units of Henrys [H].
Series: $R_{Total} = R_1 + R_2$	Series: $\frac{1}{C_{Total}} = \frac{1}{C_1} + \frac{1}{C_2}$	Series: $L_{Total} = L_1 + L_2$
Parallel: $\frac{1}{R_{Total}} = \frac{1}{R_1} + \frac{1}{R_2}$	Parallel: $C_{Total} = C_1 + C_2$	Parallel: $\frac{1}{L_{Total}} = \frac{1}{L_1} + \frac{1}{L_2}$
	The voltage across the capacitor must be continuous	The current through the inductor must be continuous
	Energy Stored $w(t) = \frac{Cv(t)^2}{2}$	Energy Stored $w(t) = \frac{Li(t)^2}{2}$

An RL Circuit Analysis

Let's consider the circuit below which consists of a voltage source, a resistor and an inductor in series. The input voltage ($v_i(t)$) is assumed to be a voltage step applied at t = 0 and the inductor current ($i_L(t)$) is assumed to be zero just before applying the input voltage. Solve the current ($i_L(t)$).





$$v_i(t) = \begin{cases} 0 \; for \; t < 0 \\ V_i \; for \; t \geq 0 \end{cases}$$

Apply KVL to the circuit above. (You can also use KCL.)

$$v_i(t) = v_R(t) + v_L(t)$$
$$v_i(t) = i_L(t)R + L\frac{di_L(t)}{dt}$$
$$\frac{di_L(t)}{dt} + \frac{1}{L/R}i_L(t) = \frac{v_i(t)}{L}$$
$$\frac{di_L(t)}{dt} + \frac{1}{\tau}i_L(t) = \frac{v_i(t)}{L}$$

given $\tau = \frac{L}{R}$ where τ is the time constant of the circuit.

We now have a first order differential equation that needs to be solved. To solve it:

1. Find the homogeneous solution (i_{LH})
2. Find the particular solution (i_{LP})
3. The total solution is then the sum of the homogeneous solution and the particular solution. Use the initial conditions to solve for the constants.

The homogeneous equation is formed by setting the driving function in the original differential equation to zero – in this case v_i is set to 0.

$$\frac{di_{LH}(t)}{dt} + \frac{1}{\tau}i_{LH}(t) = 0$$

We assume a solution of the form: $i_{LH}(t) = Ae^{st}$
Substituting this back into the homogeneous equation:

$$Ase^{st} + \frac{1}{L/R}Ae^{st} = 0$$

Assuming $A \neq 0$ and $e^{st} \neq 0$ for finite s and t, we can write the characteristic equation and solve for s.

$$s + \frac{R}{L} = 0$$
$$s = -\frac{1}{L/R} = -\frac{1}{\tau}$$

The homogeneous solution becomes: $i_{LH}(t) = Ae^{-\frac{t}{L/R}} = Ae^{-t/\tau}$

We need to find the particular solution now.

$$\frac{di_{LP}(t)}{dt} + \frac{1}{L/R}i_{LP}(t) = \frac{v_i(t)}{L}$$

Since the drive is a step, which is constant for large t, it is appropriate to assume a particular solution of the form $i_{LP} = K$ where K is a constant. Substitute back into the previous equation and take into account that $v_i(t) = V_i$ for t > 0.

$$\frac{1}{L/R}K = \frac{V_i}{L}$$
$$K = \frac{V_i}{R}$$
$$i_{LP}(t) = \frac{V_i}{R}$$

The total solution is the sum of the homogeneous and the particular solutions.

$$i_L(t) = i_{LH}(t) + i_{LP}(t)$$
$$i_L(t) = Ae^{-t/\tau} + \frac{V_i}{R}$$

To solve for A, use the initial condition, $i_L(0) = 0$.

$$i_L(0) = 0 = A + \frac{V_i}{R}$$
$$A = -\frac{V_i}{R}$$

The overall solution for $i_L(t)$ for t ≥ 0 is:

$$i_L(t) = -\frac{V_i}{R}e^{-\frac{t}{L/R}} + \frac{V_i}{R}$$
$$i_L(t) = \frac{V_i}{R}(1 - e^{-\frac{t}{L/R}})$$
$$\boldsymbol{i_L(t) = \frac{V_i}{R}(1 - e^{-\frac{t}{\tau}})}$$

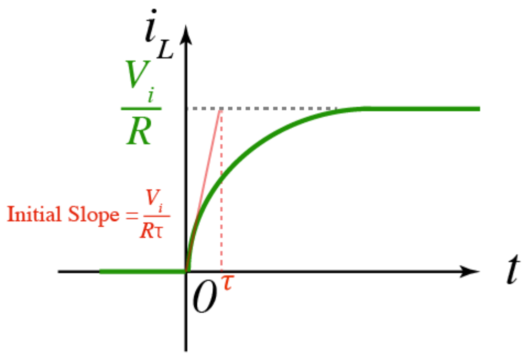
Based on $i_L(t)$ we can accordingly find $v_L(t)$ and $v_R(t)$.

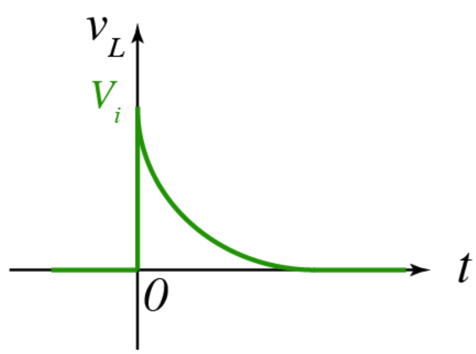
$$i_L(t) = \frac{V_i}{R}(1 - e^{-\frac{t}{\tau}})$$

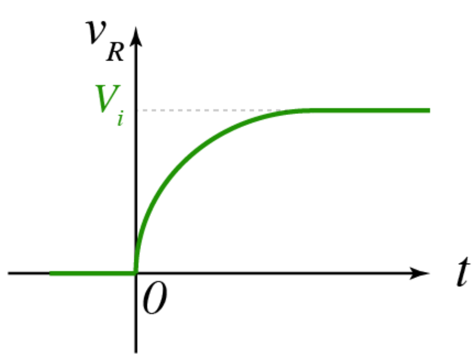
$$v_L(t) = L\frac{di_L(t)}{dt} = V_i e^{-\frac{t}{\tau}}$$

$$v_R(t) = Ri_L(t) = V_i(1 - e^{-\frac{t}{\tau}})$$

The corresponding waveforms are plotted here.







The inductor behaves like:

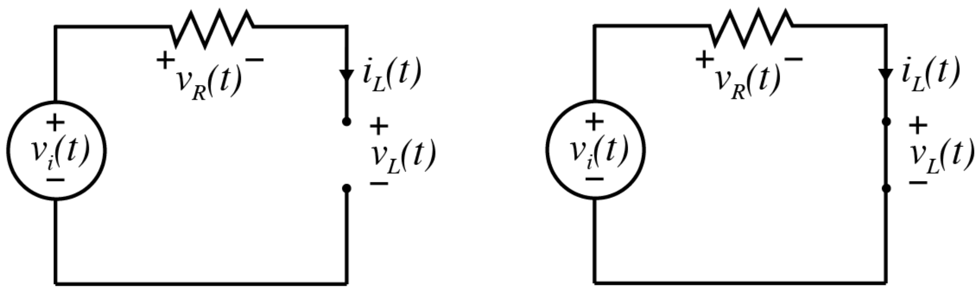
1. an instantaneous **open circuit** at t=0
2. a **short circuit** for large t

Note that this can also be solved using an <u>intuitive analysis</u> approach keeping in mind that:

$$i_L(t) = final\ value + (initial\ value - final\ value)e^{-t/time\ constant}$$

Or equivalently, this can be rearranged to:

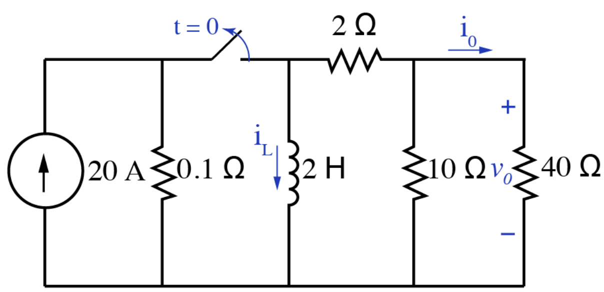
$$i_L(t) = initial\ value\ e^{-t/time\ constant} + final\ value(1 - e^{-t/time\ constant})$$



$i_L(t{=}0){=}0$ $\qquad$ $i_L(t \to \infty) = \frac{V_i}{R}$

Example of RL circuit with a switch

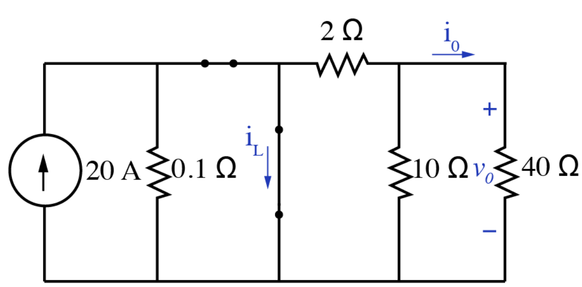
The switch in the circuit shown below has been closed for a long time before it is opened at t = 0.



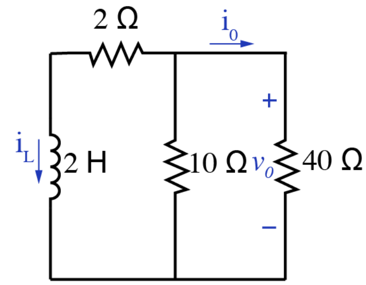
a) Find $i_L(t)\ for\ t \geq 0.$

The switch has been closed for a long time prior to t = 0, so the voltage across the inductor must be zero at $t = 0^-$, thus, $i_L(0^-) = 20A$.

For $t = 0^-$,



Given that the current through the inductor is continuous $i_L(0^+) = 20A$.
For $t \geq 0$,



$$R_{eq} = 2 + (40 \| 10) = 10\ \Omega$$
$$\tau = \frac{L}{R_{eq}} = 0.2\ s$$
$$i_L(t) = 20e^{-\frac{t}{\tau}} = 20e^{-5t}\ A$$

b) Find $i_0(t)\ for\ t \geq 0^+$.

$$i_0 = -i_L \frac{10}{10 + 40} = -0.2 i_L$$

$$i_0(t) = -0.2(20e^{-5t}) = -4e^{-5t}\ A$$

c) Find $v_0(t)\ for\ t \geq 0^+$.

$$v_0(t) = i_0(t)R = (-4e^{-5t})(40) = -160e^{-5t}\ V$$

d) Find the percentage of the total energy stored in the 2 H inductor that is dissipated in the 10 Ω resistor.

Power dissipated in 10 Ω resistor: $P_{10\Omega} = \frac{v_0^2}{R} = \frac{v_0^2}{10} = 2560e^{-10t}\ W$

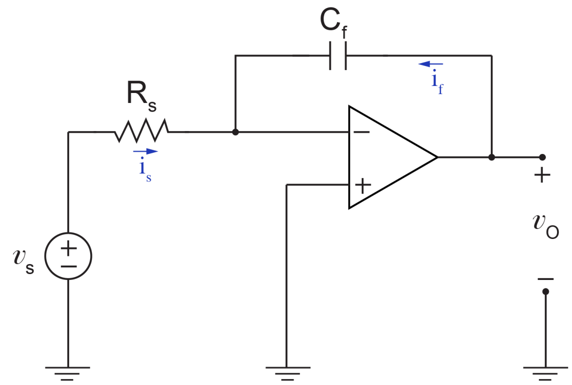
Total energy dissipated in 10 Ω resistors: $w_{10\Omega} = \int_0^\infty 2560e^{-10t}\, dt = 256\ J$

Initial energy stored in 2H inductor: $w(0) = \frac{1}{2} L i^2(0) = 400\ J$

Percentage energy dissipated in 10 Ω resistor: $\left(\frac{256}{400}\right) 100 = 64\%$

Example including an Op-Amp and a Capacitor – Integrating Amplifier

Assume that the operation amplifier is ideal. Find an expression for $v_O(t)$.



Using KCL:

$$i_s + i_f = 0$$

$$\frac{v_s}{R_s} + C_f \frac{dv_O}{dt} = 0$$

$$\frac{dv_O}{dt} = -\frac{1}{R_s C_f} v_s$$

$$v_O(t) = -\frac{1}{R_s C_f} \int_{t_0}^{t} v_s \, dt + v_O(t_0)$$

Assume $v_s(t)$ is a step response such as shown below and find $v_O(t)$.

$$v_O(t) = -\frac{1}{R_s C_f} V_m t + 0 \qquad 0 \le t \le t_1$$

$$v_O(t) = -\frac{1}{R_s C_f} \int_{t_1}^{t} (-V_m) \, dt - \frac{1}{R_s C_f} V_m t_1$$

$$= \frac{V_m}{R_s C_f} t - \frac{2V_m}{R_s C_f} t_1 \qquad t_1 \le t \le 2t_1$$