6.002 CIRCUITS AND ELECTRONICS

Lecture 7 – **Capacitors, RC Networks, Step Response**

February 27, 2020

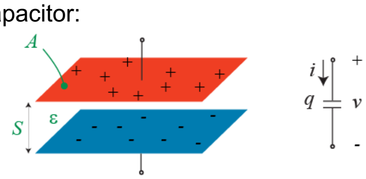
Contents:
1. Capacitor: a quick review
2. Differentiator (from Tuesday)
3. Analysis of RC circuits

Reading Assignment:
Agarwal and Lang, Ch. 9 (§ § 9.1.1, 9.2.1, 9.3.1), 10 (§ 10.1)

Handouts:
Lecture 7 notes
Quiz 1 on March 11th. It covers everything until today (including psets!)

1. Capacitor: a quick review

- Ideal parallel-plate capacitor:



- Relationship between charge in plates and voltage across capacitor [from Gauss' law]:

$$q = Cv$$

Units:

$$Coulombs = Farads \times Volts$$

In parallel plate capacitor:

$$C = \epsilon \frac{A}{S}$$

With:

$$\epsilon = permittivity\ [F/cm]$$

Device equation for capacitor

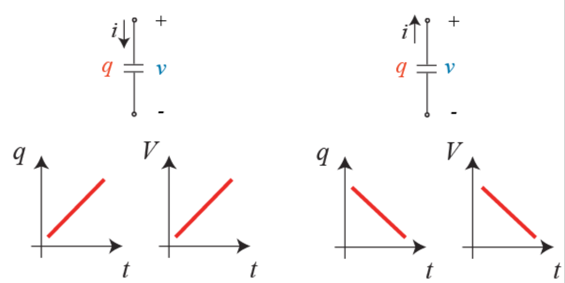
$$q = Cv$$

Current is rate of flow of charge:

$$i = \frac{dq}{dt}$$

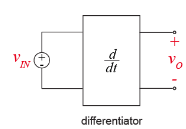
Therefore, in a capacitor:

$$i = C\frac{dv}{dt}$$

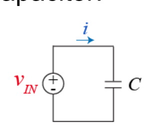


2. Differentiator (from last lecture)

This is what we want:



Could use a capacitor:

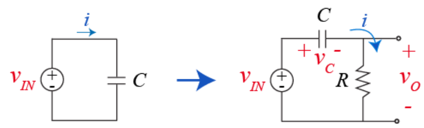


where:

$$i = C\frac{dv_{IN}}{dt}$$

– For differentiator, need to convert i into v_O

- Differentiator: use small resistance to transform i into v_O:



Want:

$$i = C\frac{dv_C}{dt} \simeq C\frac{dv_{IN}}{dt}$$

so that:

$$v_O = iR \simeq RC\frac{dv_{IN}}{dt}$$

This requires:

$$v_O \ll v_C \simeq v_{IN}$$

Or:

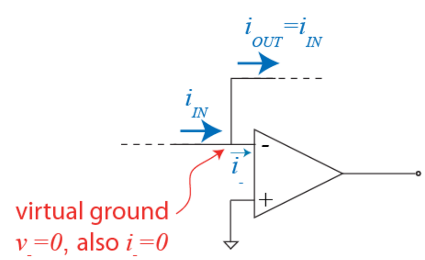
$$RC\frac{dv_{IN}}{dt} \ll v_{IN}$$

Signal time derivative can't be too high → *upper limit* to frequency content of signal

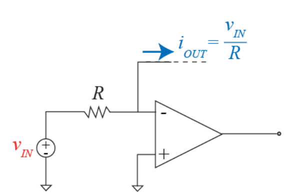
Op-amps to the rescue…

i → v and v → i transformation

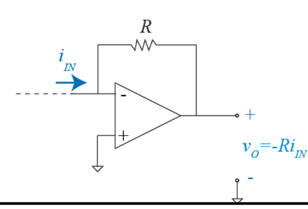
Remember this configuration:



Voltage-to-current converter:

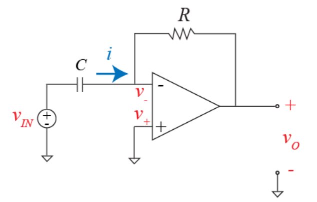


Current-to-voltage converter:



Op-amp differentiator

Use an op-amp i-v converter:



Inverting input is virtual ground:

$$v_+ = 0 \;\rightarrow\; v_- \simeq 0$$

Then

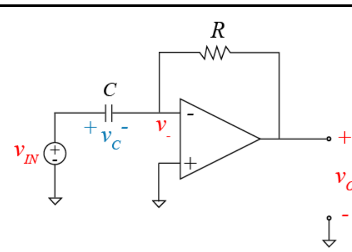
$$i \simeq C\frac{dv_{IN}}{dt}$$

At output:

$$v_O = -iR$$

All together:

$$v_O \simeq -RC\frac{dv_{IN}}{dt}$$



Differentiator is accurate if:

$$v_C \simeq v_{IN}$$

which requires:

$$|v_-| \ll v_{IN}$$

v_- is given by:

$$v_+ - v_- = \frac{v_O}{A} \;\rightarrow\; v_- = -\frac{v_O}{A} = -\frac{RC}{A}\frac{dv_{IN}}{dt}$$

We need now:

$$|\frac{RC}{A}\frac{dv_{IN}}{dt}| << v_{IN}$$

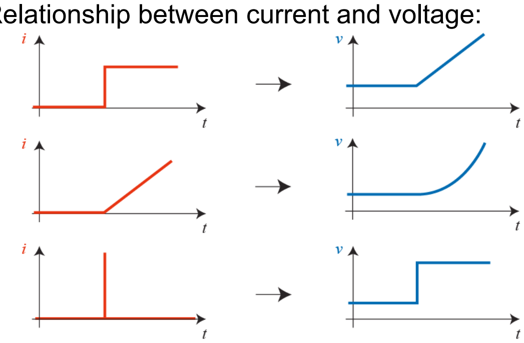
Bandwidth of differentiator greatly improved!

Coming back to capacitors…

- Another way to write capacitor equation:

$$i = C\frac{dv}{dt} \rightarrow v = \frac{1}{C}\int_{-\infty}^{t} i\, dt$$

- Relationship between current and voltage:



We need initial conditions to fully define the state of the capacitor

Abrupt change in voltage gives rise to infinite current! → Not possible in real circuits!

Voltage across capacitor depends on the entire past history of current flowing through it → memory!

Power and energy in a capacitor

- Power is rate of flow of energy. For capacitor:

$$p = vi = vC\frac{dv}{dt} = \frac{d}{dt}(\frac{1}{2}Cv^2) = \frac{dw_E}{dt}$$

  - If $dv^2/dt>0$ → $p>0$: energy being dumped into C
  - If $dv^2/dt<0$ → $p<0$: energy being removed from C

- Unlike resistor, capacitor stores energy (in electrostatic form).
- Electrostatic energy stored in capacitor:

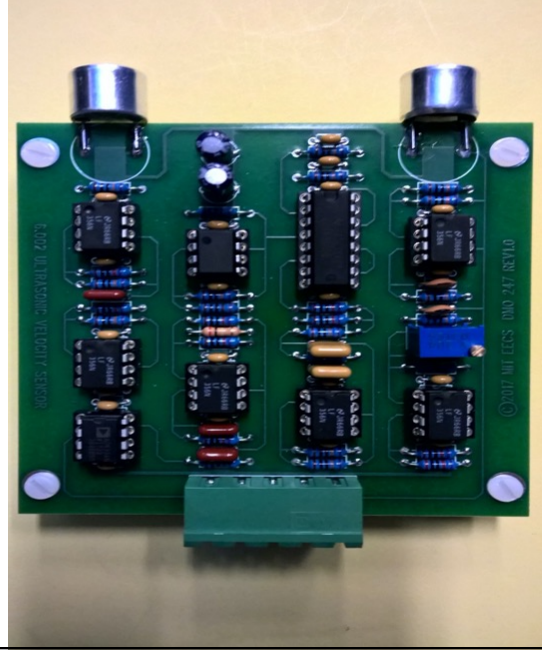
$$w_E = \frac{1}{2}Cv^2$$

- Capacitors can store substantial amounts of energy!

Demo

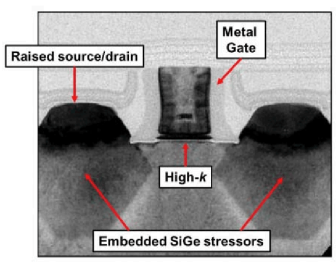
Capacitors are everywhere…

Doppler Ultrasound System

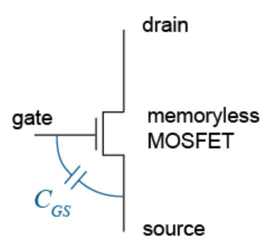


Capacitors are there, even when you don't see them…

- Transistors (MOSFETs) really looks like:



Intel's 32 nm MOSFET



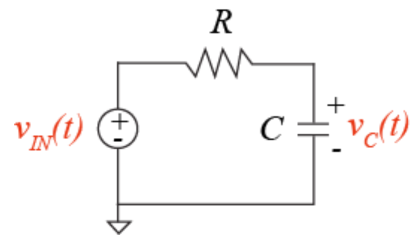
- The Capacitance between the gate and the source is key to give the transistor its ability to control current flow (and its time response).

- All other devices have capacitance associated with them.

→ Need to understand impact of capacitors on circuit operation

3. Analysis of RC circuits

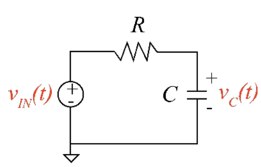
- Consider the following circuit:

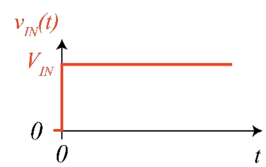


- Circuit behavior completely determined by:
  - Differential equation describing circuit
  - Input waveforms
  - Initial conditions
- Consider example with:
  - Input step function: $v_{IN}(t)=V_{IN}\,u_o(t)$
  - Initial condition: $v_C(t=0^-)=0$

- How does $v_c(t)$ evolve in time?

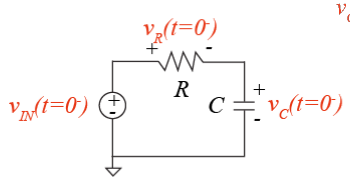
- First, **think physics**!





$v_C(t=0^-)=0$

- Situation at t=0⁻:



At $t=0^-$:

$$v_{IN}(t=0^-)=0$$

$$v_C(t=0^-)=0$$

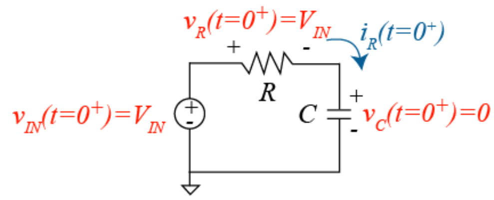
Since KVL demands that:

$$v_{IN}=v_R+v_C$$

Then:

$$v_R(t=0^-)=0$$

- Situation at t=0$^+$:



Capacitor can't change voltage abruptly:

$$v_C(t=0^+)=0$$

V_{IN} appears across R:

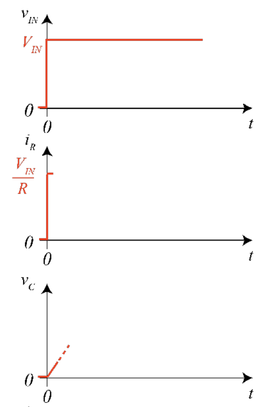
$$v_R(t=0^+)=V_{IN}$$

Resistor current:

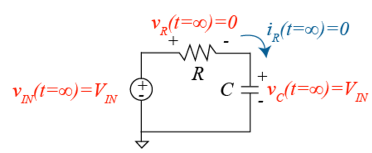
$$i_R(t=0^+)=\frac{V_{IN}}{R}$$

i_R starts charging C:

$$\frac{dv_C}{dt}\Big|_{t=0^+}>0$$



- Situation at t=∞:



Capacitor will charge until it reaches the maximum voltage possible, V_{IN}:

$$v_C(t=\infty)=V_{IN}$$

No voltage across R:

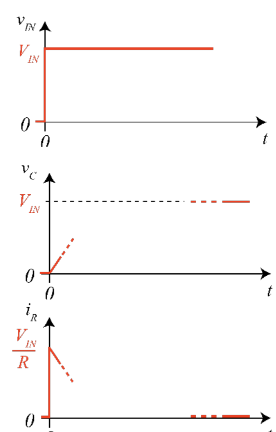
$$v_R(t=\infty)=0$$

Resistor current is zero:

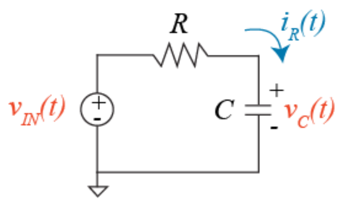
$$i_R(t=\infty)=0$$

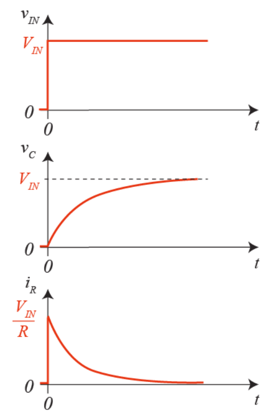
Nothing changes from here on:

→ *steady-state condition*



- Solution roughly looks like:



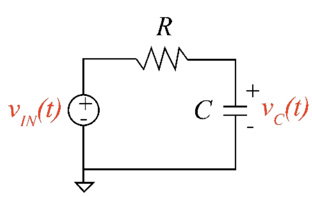


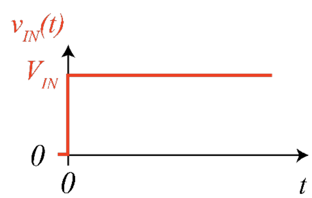
- Features of solution:
  - Increase of $v_C(t)$ slows down with time
  - Given enough time, $v_C(t)$ saturates
  - $i_R(t)$ peaks at $t=0^+$ and drops from there to eventually zero

- What is the detailed shape? How long does it take for steady-state to be achieved? → need to solve problem mathematically

Mathematical solution (18.03 to the rescue!)

- Given:



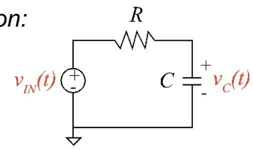


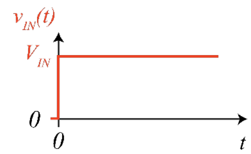
$v_C(t=0^-)=0$

- Want $v_C(t)$

- Steps:
  1. Formulate differential equation
  2. Find particular solution
  3. Find homogeneous solution
  4. Apply initial conditions

1. *Formulate differential equation:*





$v_C(t=0^-)=0$

One node analysis.
Node equation for $v_C(t)$:

$$\frac{v_C(t) - v_{IN}(t)}{R} + C\frac{dv_C(t)}{dt} = 0$$

Rearrange:

$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_{IN}(t)$$

Solution of form:

$$v_C(t) = v_{CH}(t) + v_{CP}(t)$$

Homogeneous solution ("natural response")

Particular solution ("forced response")

2. *Find particular solution:*

Interested in solution for $t>0$. In this range:

$$v_{IN}(t) = V_{IN}$$

Differential equation becomes:

$$RC\frac{dv_C(t)}{dt} + v_C(t) = V_{IN}$$

Particular solution satisfies differential equation:

$$RC\frac{dv_{CP}(t)}{dt} + v_{CP}(t) = V_{IN}$$

Try solution of form:

$$v_{CP}(t) = K$$

Then:

$$v_{CP}(t) = V_{IN}$$

3. Find homogeneous solution

Homogeneous differential equation:

$$RC\frac{dv_{CH}(t)}{dt} + v_{CH}(t) = 0$$

Solution of the form:

$$v_{CH}(t) = Ae^{-\frac{t}{\tau}}$$

Value of τ that satisfies differential equation:

$$\tau = RC$$

Then:

$$v_{CH}(t) = Ae^{-\frac{t}{RC}}$$

4. Apply initial conditions

Total solution so far:

$$v_C(t) = V_{IN} + Ae^{-\frac{t}{RC}}$$

Initial condition:

$$v_C(0) = 0$$

[voltage across capacitor can't change abruptly]

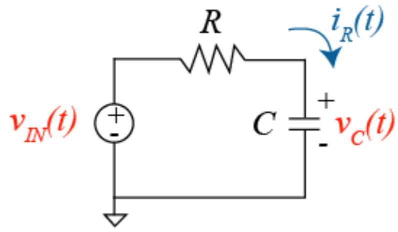
Then:

$$A = -V_{IN}$$

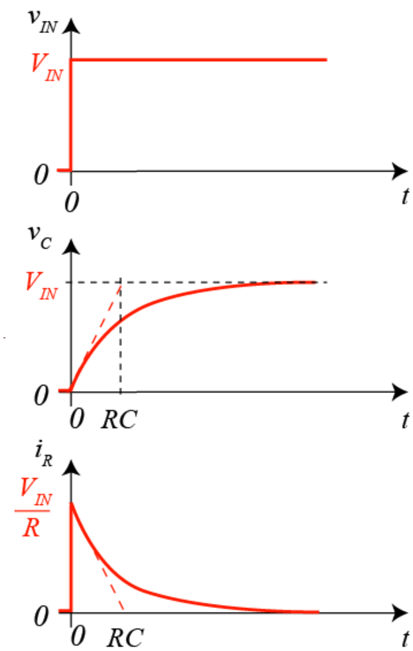
Final solution:

$$v_C(t) = V_{IN}(1 - e^{-\frac{t}{RC}})$$

- Can also compute current through R:



$$i_R(t) = \frac{v_{IN}(t) - v_C(t)}{R} = \frac{V_{IN}}{R} e^{-\frac{t}{RC}}$$



Summary

- Capacitors are everywhere… (even when you don't see them)
- Circuits with capacitors exhibit memory: voltage across capacitor depends on the entire past history of current flowing through it.
- A MOSFET has capacitance associated with its gate.
- Device equation for capacitor:

$$i = C\frac{dv}{dt}$$

- Capacitors can store energy.
- RC circuits characterized by first-order linear differential equation.
- Time constant of RC circuits: $\tau = RC$.