Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Circuits and Electronics
Fall 2016

Quiz 3 Practice Problems

Circuits for Exercises Use the circuits shown in figure 1 below for exercises 1 through 5.

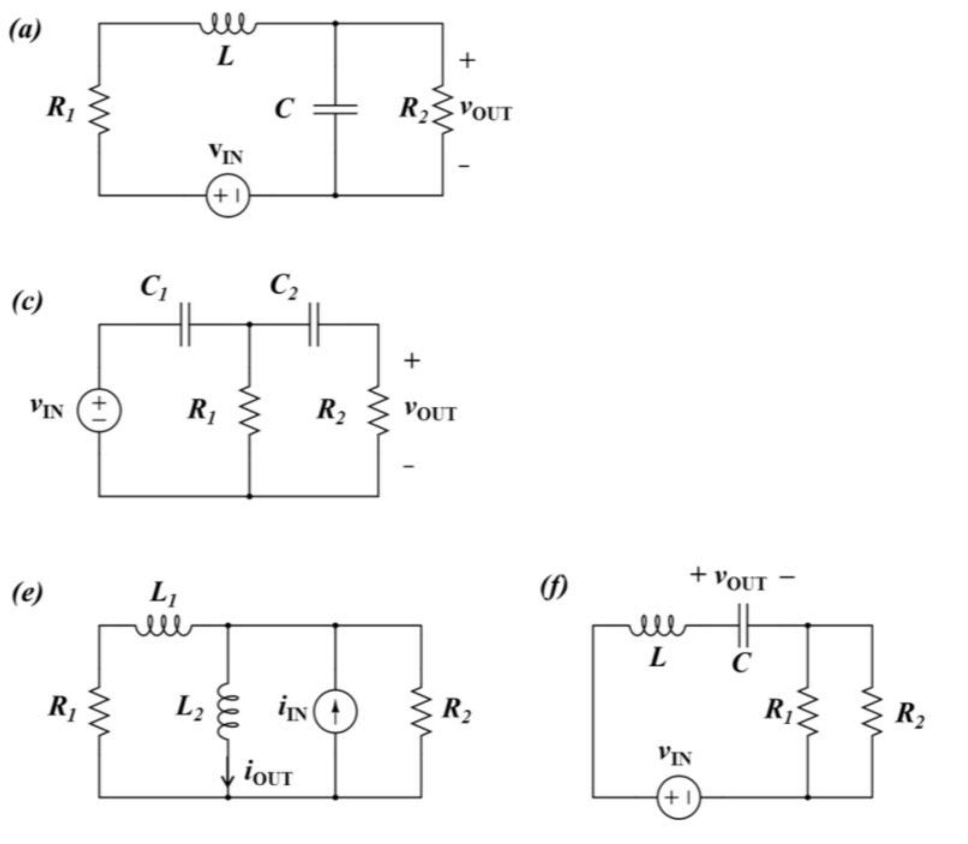


Figure 1

Exercise 1 Find the equation of motion for each circuit in figure 1 in terms of the indicated variables. You may assume all input voltage and current sources and all circuit element variables are given.

Exercise 2 Write the transfer function $H(j\omega)$ in the form $\frac{(j\omega - S_{Z1})((j\omega - S_{Z2})...}{(j\omega - S_{P1})((j\omega - S_{P2})...}$.

(a) Find $H(j\omega)$ for circuit (a) in figure 1.

(b) Find $H(j\omega)$ for circuit (c) in figure 1.

(c) Find $H(j\omega)$ for circuit (e) in figure 1.

(d) Find $H(j\omega)$ for circuit (f) in figure 1.

Exercise 3 For each of the circuits in figure 1, assume a generic step function drive. That is, assume $v_{IN} = u(t)V_{IN}$ and $i_{IN} = u(t)I_{IN}$.

(a) List the initial current and voltage in the reactive circuit elements before and after the input step.

(b) Use your answers to part (a) to establish the requisite initial conditions to solve the differential equation.

(c) Solve for v_{OUT} for circuit (a) in figure 1.

(d) Solve for v_{OUT} for circuit (f) in figure 1.

Exercise 4 Assume for each circuit in figure 1 that the drive is of the form $Acos(\omega t + \phi)$. Determine the indicated output variable.

Exercise 5 For each of the circuits in figure 1, replace the sources with impulses. That is, assume $i_{IN} = Q\delta(t)$ and $v_{IN} = \Lambda\delta(t)$. Write the expression for the indicated output variable vs time for $t > 0$.

Problem 1 In one of the in-class demos, we showed how a long cable could contribute enough capacitance to distort and filter a waveform. Scope probes are designed to address this problem by using a capacitive voltage divider composed of the variable capacitor C and the 8pF capacitor shown below in figure 2. C_{cable} models the added capacitance contributed by the cable in the demo.

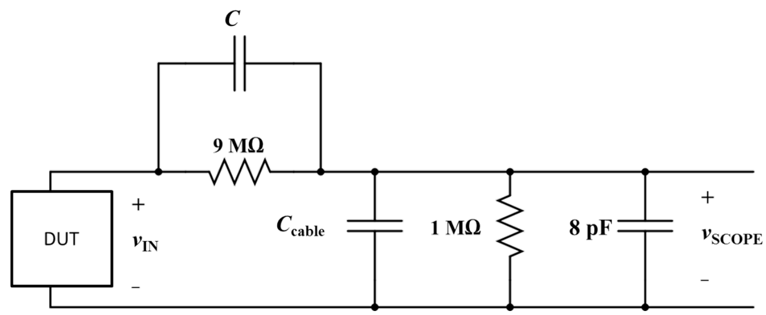


Figure 2: Problem 9.1

The experiment uses a $3ft$ long 50Ω coax cable (Here, the 50Ω refers to the transmission line impedance. In case you are interested, that is the ratio of the amplitude of v_{IN} to i_{IN} when the cable is driven by high frequency sinusoidal input voltage v_{IN}.) However, the 50Ω impedance is not relevant for this problem. The only point of relevance is the cable parasitic capacitance, which is $30\frac{pF}{ft}$. Thus the $3ft$ long coax cable adds $90pF$ of capacitance in parallel with the $8pF$ scope capacitance.

(a) What should C be such that $H(j\omega) = \frac{v_{SCOPE}(j\omega)}{v_{IN}(j\omega)}$ has no frequency dependence? What is the resulting $H(j\omega)$ with the chosen C?

(b) Prove that this circuit will faithfully reproduce any input waveform, v_{IN} at the scope, i.e. that $v_{SCOPE}(t) = Av_{IN}(t)$ for any v_{IN} and a constant A.

(c) You have a new kind of scope that plots current,i_{SCOPE} rather than voltage,v_{SCOPE}, at its output port. Design a probe that will faithfully reproduce the shape of $i_{IN}(t)$ on the scope, removing the filtering influence of C_{scope}.

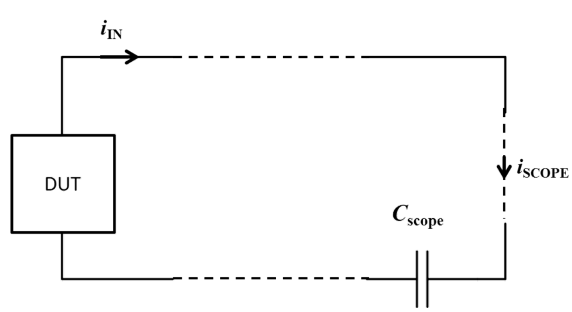


Figure 3: Problem 1

Problem 2 Consider an unknown circuit shown in figure 4 , exhibiting the frequency response shown in figure 5.

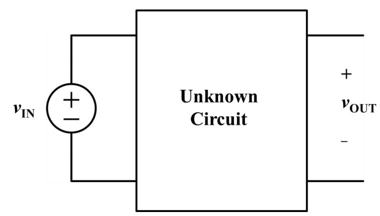


Figure 4: Problem 2

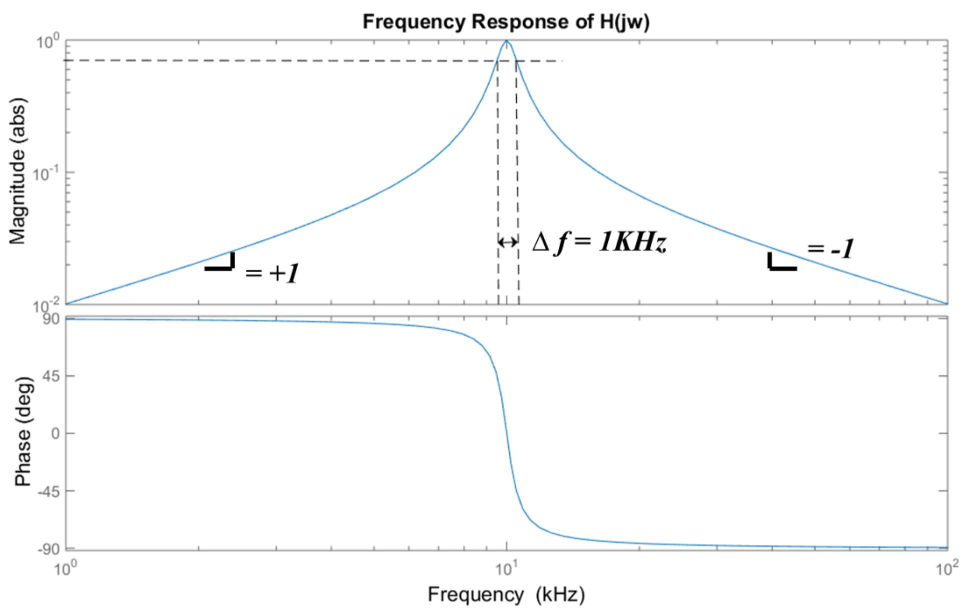


Figure 5: Problem 2

(a) What is the radial resonance frequency ω_o?

(b) What is the quality factor Q for this filter?

(c) Assuming $v_{IN} = 3V * cos(2\pi * 9kHz + 0.3)$, determine v_{OUT}.

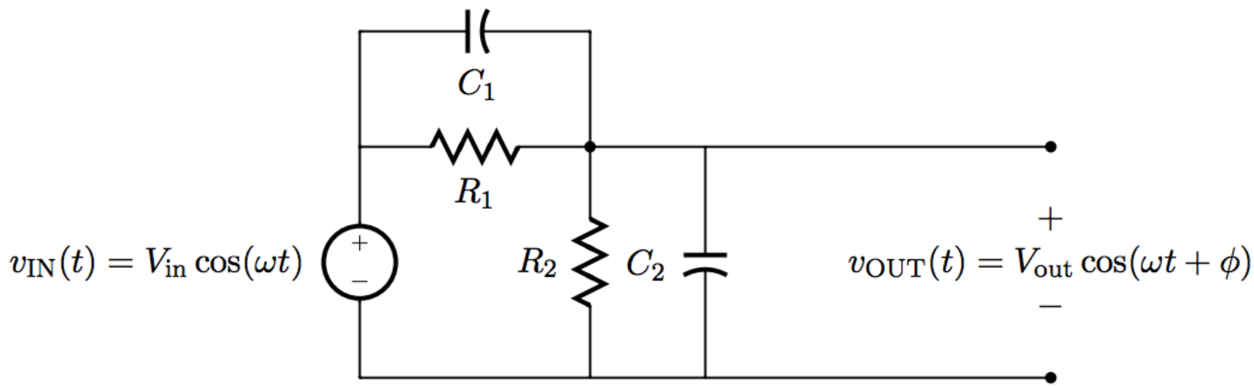
(d) Assuming $v_{IN} = 3V * (1 - u(t))$, determine the damping factor α and the period T between zero-crossings of v_{OUT} for $t > 0$. $T = \frac{2\pi}{\omega_d}$ where $\omega_d = \sqrt{\omega_o^2 - \alpha^2}$.

(e) Suggest an LRC circuit for the black box, choosing L,R,and C values appropriately.

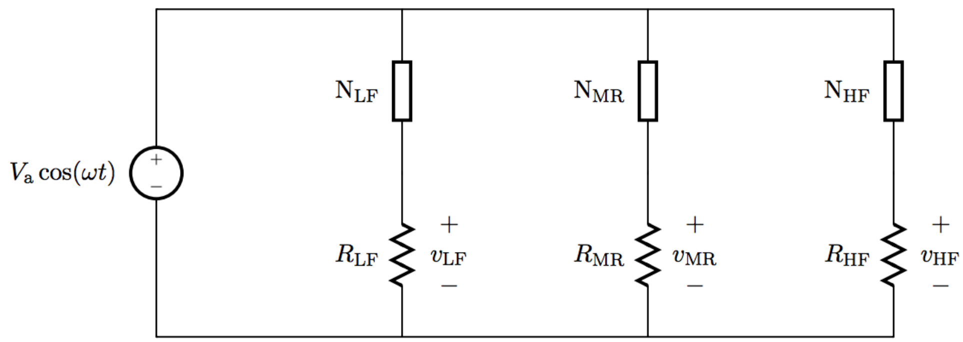
(f) For the circuit from part(e), find $v_{OUT}(0 + \varepsilon)$ and $\frac{d}{dt}v_{OUT}(0 + \varepsilon)$, the initial conditions.

(g) Using the standard form of a step response to an LRC circuit, write an analytic expression for $v_{OUT}(t)$ for $t > 0$.

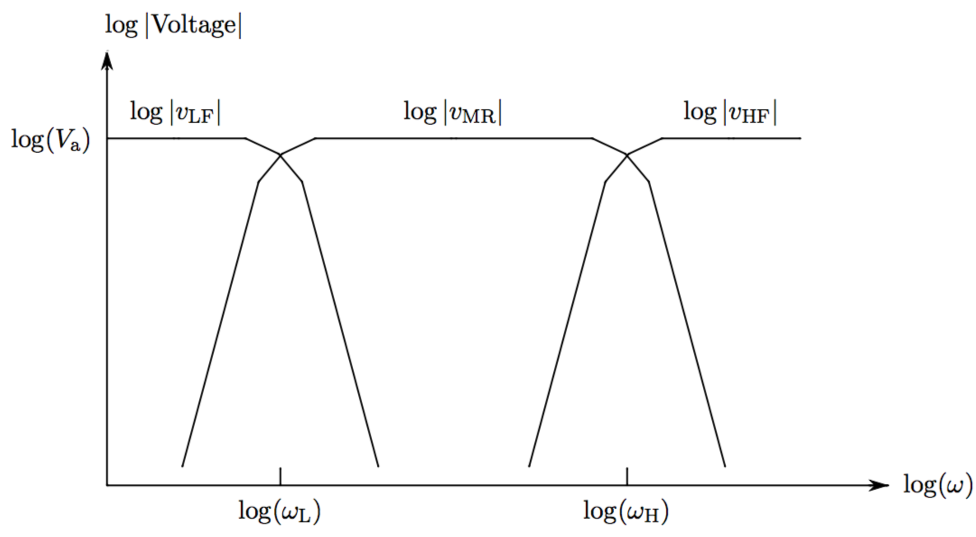
Problem 3 The network shown below models an oscilloscope probe that provides a 10:1 voltage attenuation. Resistor R_1 is a fixed resistor in the probe, resistor R_2 models the input resistance of the oscilloscope, capacitor C_1 is a variable capacitor in the probe, and capacitor C_2 models the combined input capacitance of the oscilloscope and the cable between the probe and the oscilloscope. What relations are required between R_1, R_2, C_1 and C_2 so that $v_{\text{OUT}}(t) = 0.1 v_{\text{IN}}(t)$ for all ω? That is, what relations are required so that $V_{\text{out}} = 0.1 V_{in}$ and $\phi = 0$ for all ω? (Note that the value of C_2 is difficult to guarantee in practice due to variations in cable length and oscilloscope input capacitance, so C_1 is made manually adjustable in the probe.)



Problem 4 This problem focuses on the (greatly simplified) design of crossover networks for audio speaker systems driven by a single amplifier. The purpose of these networks is to direct low-frequency signals to a low frequency (LF) speaker, mid-range signals to a mid-range (MR) speaker, and high-frequency signals to a high-frequency (HF) speaker. The interconnection of the amplifier, modeled here as a cosinusoidal voltage source, the three speakers, each modeled here as a resistor, and the three cross-over networks is shown below. *Note that this was once a final exam problem.*



The three cross-over networks N_{LF}, N_{MR}, and N_{HF} may each be a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor. The topology of each network, and the values of the components in each network, are to be designed to provide the three voltage responses shown below as functions of frequency. Note that ω_{L} and ω_{H} are the frequencies at which $|v_{\mathrm{LF}}|$, $|v_{\mathrm{MR}}|$ and $|v_{\mathrm{HF}}|$ fall to the value of $V_a/\sqrt{2}$. They are related by $\omega_{\mathrm{L}} \ll \omega_{\mathrm{H}}$ in this problem.



(**a**) Consider the network N_{LF} for the low-frequency speaker. What type of network should be used for N_{LF}: a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor?

(**b**) Determine the values of the corresponding capacitance C_{LF}, if used, and inductance L_{LF}, if used, in terms of R_{LF}, R_{MR}, R_{HF}, ω_{L} and ω_{H}.

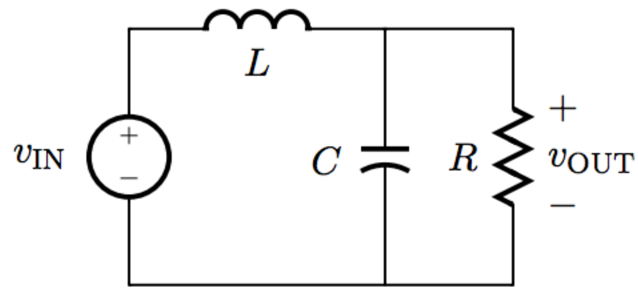
(c) Consider the network N_{MR} for the mid-range speaker. What type of network should be used for N_{MR}, a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor?

(d) Determine the values of the corresponding capacitance C_{MR}, if used, and inductance L_{MR}, if used, in terms of R_{LF}, R_{MR}, R_{HF}, ω_{L} and ω_{H}.

(e) Consider the network N_{HF} for the high-frequency speaker. What type of network should be used for N_{HF}, a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor?

(f) Determine the values of the corresponding capacitance C_{HF}, if used, and inductance L_{HF}, if used, in terms of R_{LF}, R_{MR}, R_{HF}, ω_{L} and ω_{H}.

Problem 5 In this problem, a low-voltage sinusoidal source is coupled to a resistive load through an inductor-capacitor network as shown below. The role of the network is to boost the voltage at the load.



(**a**) Derive a second-order differential equation that describes the evolution of v_{OUT} as driven by v_{IN}. You need not solve the differential equation.

(**b**) Assume that the circuit operates in the sinusoidal steady state with $v_{IN}(t) = V_{in}cos(\omega t)$. Correspondingly, let $v_{OUT}(t)$ take the form $v_{OUT} = V_{out}cos(\omega t + \phi)$. Determine ϕ, and the voltage gain G defined by $G \equiv V_{out}/V_{in}$.

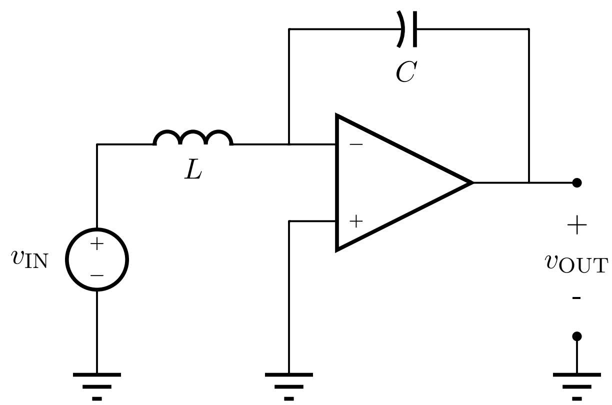
(**c**) For a given R, L and ω, determine the value of C that maximizes G, and for this value of C, determine G.

(**d**) Suppose that $v_{IN}(t)$ is abruptly set to zero in an attempt to remove the voltage at the load. In this case, the amplitude of the load voltage will decay in proportion to $e^{t/\tau}$. Assuming that C is chosen to maximize G following the result from Part (**c**), determine τ in terms of G and ω. Hint: can you get the time constant (as a function of C, L and R) from the differential equation found in Part (**a**)?

(**e**) In view of the results of Parts (**c**) and (**d**), what is the disadvantage of using an inductor-capacitor network to boost the voltage which excites the load?

Problem 2: Op-Amps – 20%

Assume that the op-amps in this problem are ideal. For Parts (2A) and (2B), consider the circuit shown below.



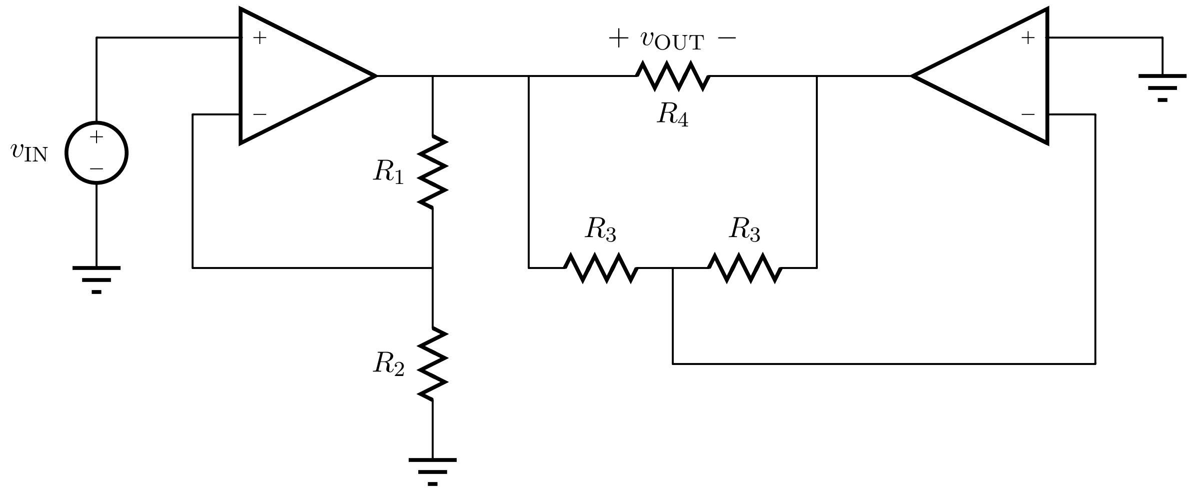
(A) Assume that the inductor and capacitor are initially at rest. At $t = 0$ the voltage source steps to V such that $v_{\text{IN}}(t) = Vu(t)$ where $u(t)$ is the unit step function. Determine v_{OUT}(t) for $t \geq 0$.

$v_{\text{OUT}}(t) =$

(B) Let $v_{\mathrm{IN}}(t) = V\cos(\omega t)$. Further, assume that v_{OUT} has a zero-valued time average. Determine v_{OUT} in the sinusoidal steady state.

$v_{\mathrm{OUT}}(t) =$

For Part (2C), consider the circuit shown below.

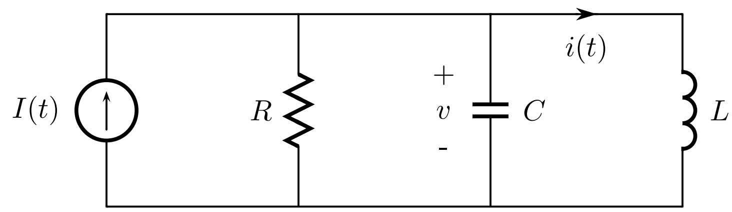


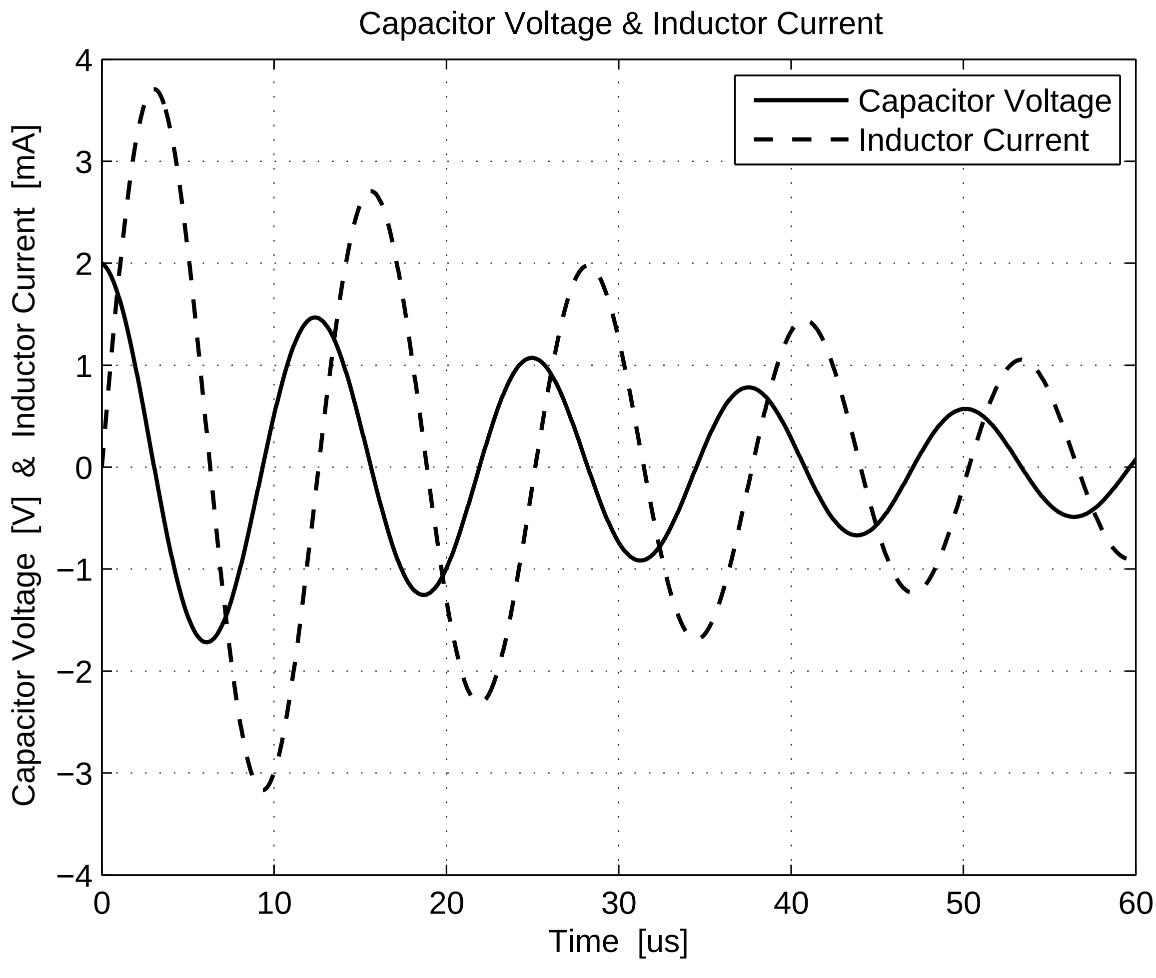
(C) Determine the output voltage v_{OUT}.

$v_{\mathrm{OUT}} =$

Problem 3: Second-Order Network Transients – 20%

This problem concerns the second-order network shown below. To begin, the current source delivers a current impulse with area Q, the total charge delivered, such that $I(t) = Q\delta(t)$. A graph of the subsequent capacitor voltage v in volts and inductor current i in milliamps is also shown below as a function of time in microseconds. Use the graphical waveforms to answer the following questions.





(A) What is the *approximate* value of the inductance L? *A numerical answer with proper units is expected.*

$L =$

(B) What is the *approximate* value of the capacitance C? *A numerical answer with proper units is expected.*

$C =$

(C) What is the *approximate* value of the resistance R? Hint: $e^{-0.5} \approx 0.6$. *A numerical answer with proper units is expected.*

$R =$

(D) What is the *approximate* value of the impulse charge Q? *A numerical answer with proper units is expected.*

Charge $Q =$

(E) What is the *approximate* value of the quality factor of the network? *A numerical answer with proper units is expected.*

Quality Factor =

(F) Complete each sentence in the chart below by circling the phrase that correctly replaces the question marks. In completing the last two sentences, assume that the impulse charge Q remains fixed.

The oscillation frequency ??? with increasing L.	increases decreases stays fixed
The oscillation frequency ??? with increasing C.	increases decreases stays fixed
The decay rate ??? with increasing L.	increases decreases stays fixed
The decay rate ??? with increasing C.	increases decreases stays fixed
The decay rate ??? with increasing R.	increases decreases stays fixed
The current amplitude ??? with increasing L.	increases decreases stays fixed
The voltage amplitude ??? with increasing L.	increases decreases stays fixed

Finally, suppose that the current source is a sinusoid taking the form $I(t) = I_\circ \cos(\omega t)$, and that values of R, C, and L are unchanged. In this case, answer the following questions assuming that the network operates in the sinusoidal steady state.

(G) For a given value of $I_\circ$, *approximately* what value for the frequency ω will yield the largest amplitude for the voltage v? *A numerical answer with proper units is expected.*

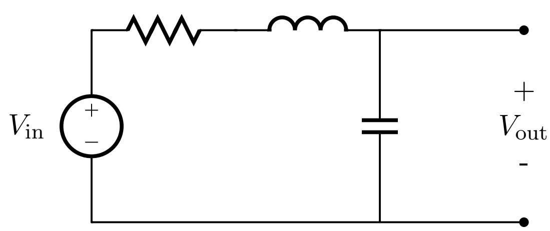
$\omega =$

(H) Assume that ω is set to be that frequency found in Part (3G) and that $I_\circ = 1$ mA. What will be the *approximate* maximized amplitude of v? *A numerical answer with proper units is expected.*

$|v| =$

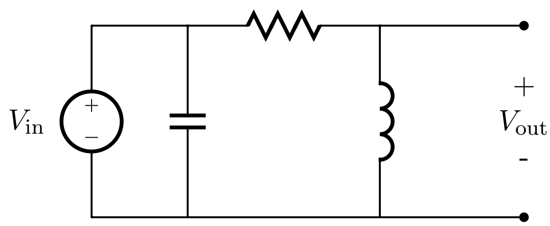
Problem 4: Sinusoidal Steady State – 20%

The circuits shown below are driven by sinusoidal voltage sources, and operate in the sinusoidal steady state with transfer function $H(\omega) \equiv V_{\text{out}}/V_{\text{in}}$ where V_{in} and V_{out} are complex voltage magnitudes. For each circuit below, circle the letter that corresponds to the plot of its transfer function magnitude (A-F), and to the plot of its transfer function phase (G-L); the plots are shown on the following page. The magnitude plots have log-log scales, and the phase plots have log-linear scales. These plots are sketches, and so you should choose one with a correct approximate shape. Further, the horizontal axes of the log-magnitude plots do not necessarily intersect the vertical axes at $\log|H| = 1$. The op-amp in the last circuit below is ideal.



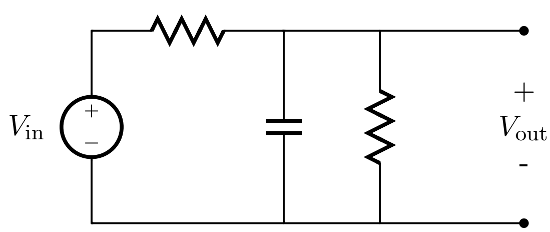
Magnitude : A B C D E F

Phase : G H I J K L



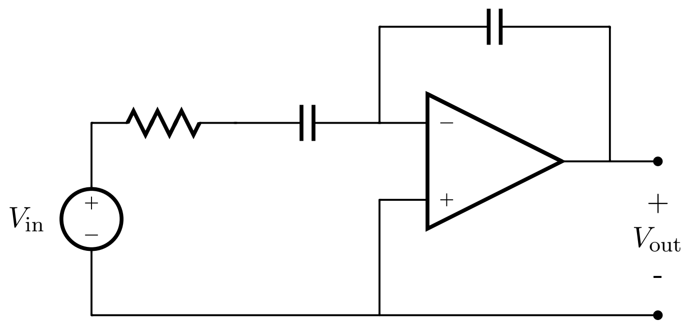
Magnitude : A B C D E F

Phase : G H I J K L



Magnitude : A B C D E F

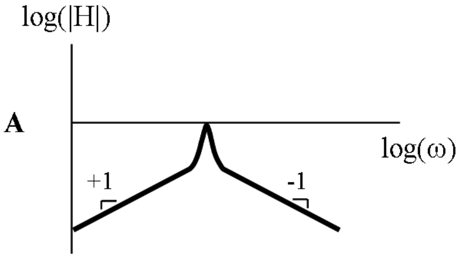
Phase : G H I J K L



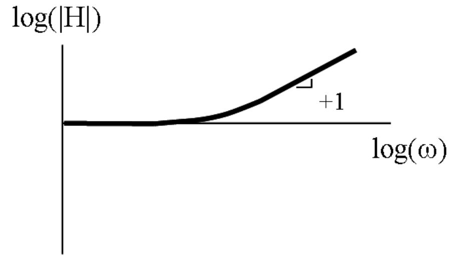
Magnitude : A B C D E F

Phase : G H I J K L

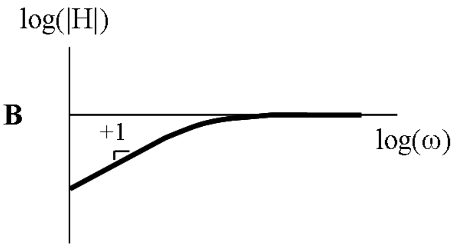
A



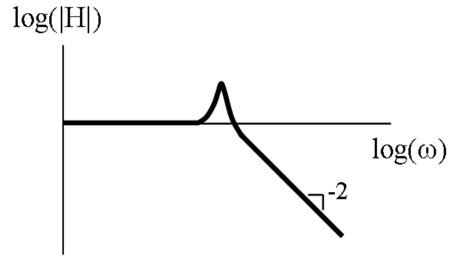
D



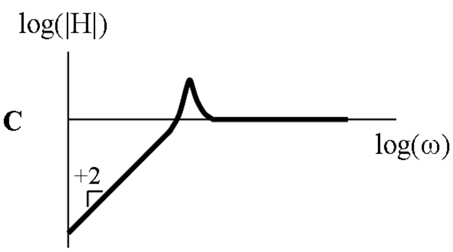
B



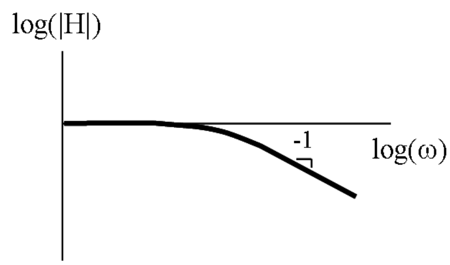
E



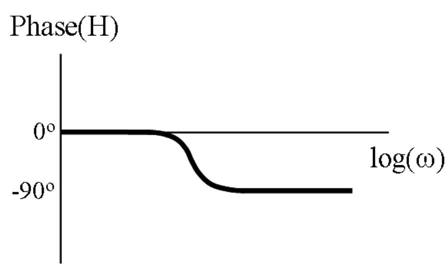
C



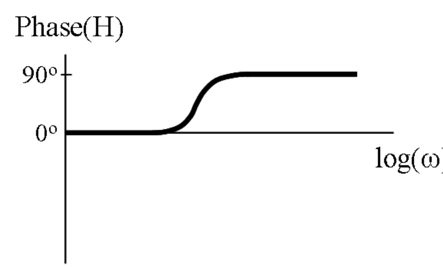
F



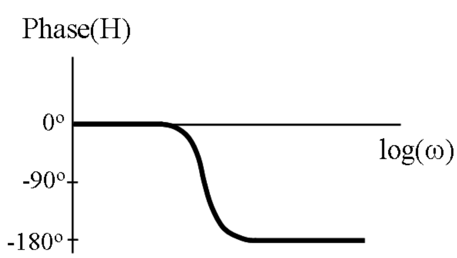
G



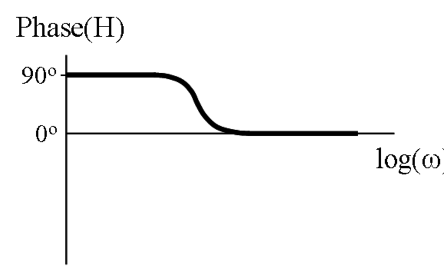
J



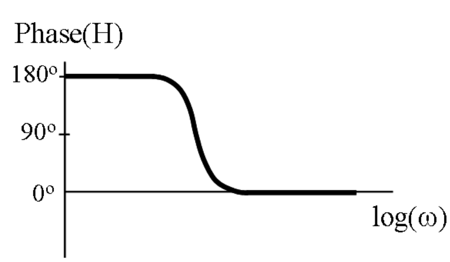
H



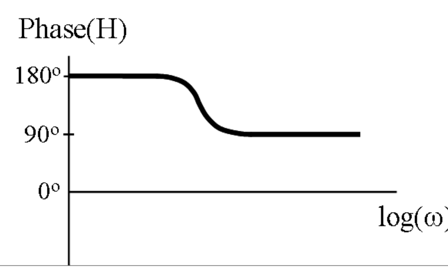
K



I



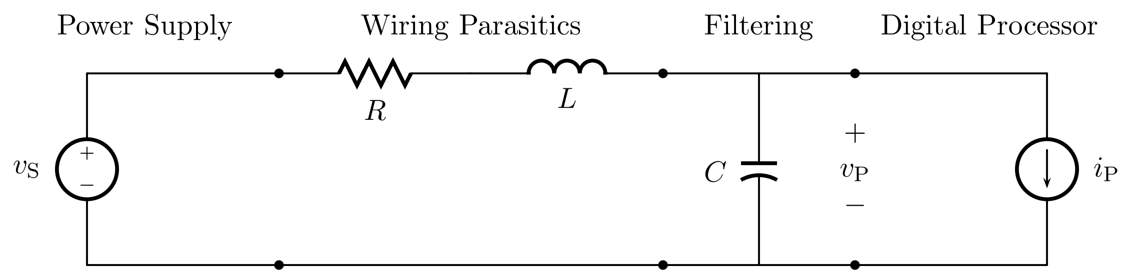
L



Problem 2 – 35%

The circuit shown below models the electrical interaction between a digital processor, its power supply, and the connection between the two. Specifically, the current source (i_P) models the processor, the voltage source (v_S) models the supply, the resistor (R) and inductor (L) model the parasitics introduced by the interconnect wiring, and the capacitor (C) helps filter the processor voltage (v_P).

Both i_P and v_S can be functions of time. For example, the current drawn by the processor will depend on the number of its sections that are active, and the speed at which they operate, all of which can vary dynamically. Similarly, the supply voltage can vary due to external disturbances. Variations in i_P and v_S will in turn cause variations in v_P, which are important to understand because the processor will not operate properly if the variations in v_P are too large.



(2A) (5%) Derive a second-order differential equation that relates v_P to v_S and i_P.

Diff Eqn:

(2B) (10%) Assume that the supply voltage v_S and the processor current i_P have both been constant for a long time at the values V_S and I_{P1}, respectively. Then, at $t = 0$, i_P takes a step such that $i_P(t) = I_{P1} + I_{P2}u(t)$. In response, for $t \geq 0$, v_P takes the form $v_P(t) = V_{P1} + V_{P2}e^{-\alpha t}\cos(\omega t + \phi)$. Determine the constants V_{P1}, V_{P2}, α, ω and ϕ in terms of R, L, C and the source parameters; you may also express V_{P1}, V_{P2}, α, ω and ϕ in terms of each other as long as only simple back substitution is required to ultimately express them in terms of R, L, C and the source parameters.

$V_{P1} =$	$V_{P2} =$
$\alpha =$	$\omega =$
$\phi =$	

(2C) (10%) Assume that the supply voltage v_{S} and the processor current i_{P} have both been constant for a long time at the values V_{S} and I_{P}, respectively. Then, at $t = 0$, a noise spike occurs in v_{S} such that $v_{\mathrm{S}} = V_{\mathrm{S}} + \Lambda\delta(t)$, where Λ is a constant. Determine the values of v_{P} and dv_{P}/dt just after the noise spike in terms of R, L, C and the source parameters.

$v_{\mathrm{P}}(0^+) =$	$dv_{\mathrm{P}}/dt(0^+) =$

(2D) (10%) A cyclic program is written for the processor that results in $i_{\mathrm{P}} = I_{\mathrm{P}} + I_{\mathrm{C}} \cos(\omega t)$, while $v_{\mathrm{S}} = V_{\mathrm{S}}$, where I_{P}, I_{C} and V_{S} are all constants. The steady-state response to the cyclic processor current takes the form $v_{\mathrm{P}} = V_{\mathrm{P}} + V_{\mathrm{C}} \cos(\omega t + \phi)$. Determine the constants V_{P}, V_{C} and ϕ in terms of R, L, C, ω and the source parameters.

$V_{\mathrm{P}} =$

$V_{\mathrm{C}} =$

$\phi =$

Problem 3 – 25%

The circuit shown below is initially at rest such that $v_C(0^-) = 0$ and $i_L(0^-) = 0$. At $t = 0$, the current source steps up to produce 1 A. The figures shown below plot four waveforms within the circuit following the current step. Their horizontal axes all show time measured in seconds [s]. Their vertical axes show either voltage measured in Volts [V] or current measured in Amperes [A], as appropriate. *Note that the initial slopes of the waveforms in Figures 1 and 3 are positive, while the initial slopes of the waveforms in Figures 2 and 4 are zero.*

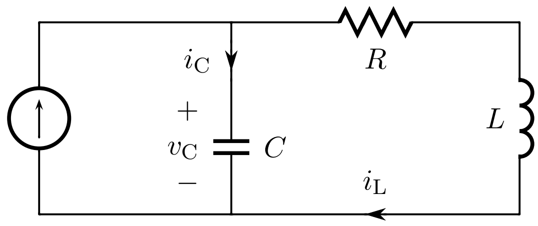


Figure 1

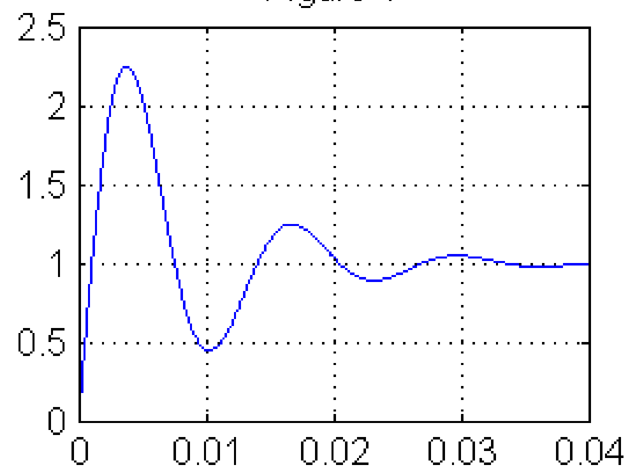


Figure 2

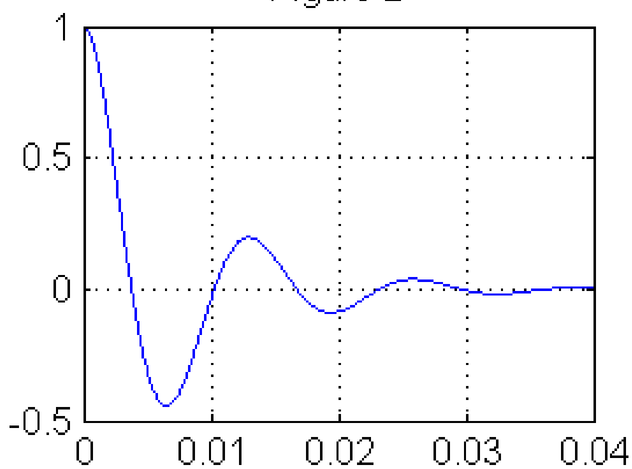


Figure 3

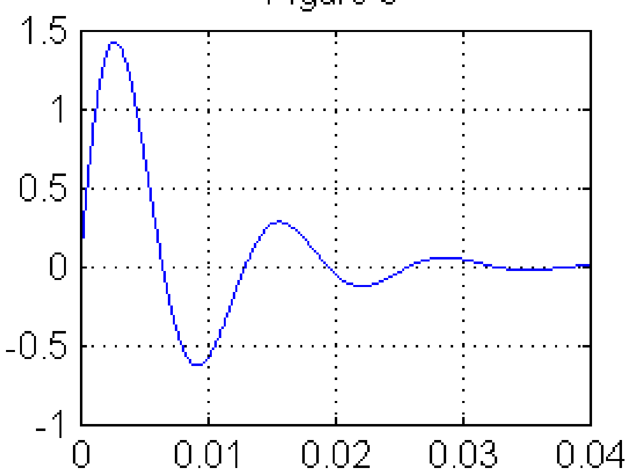
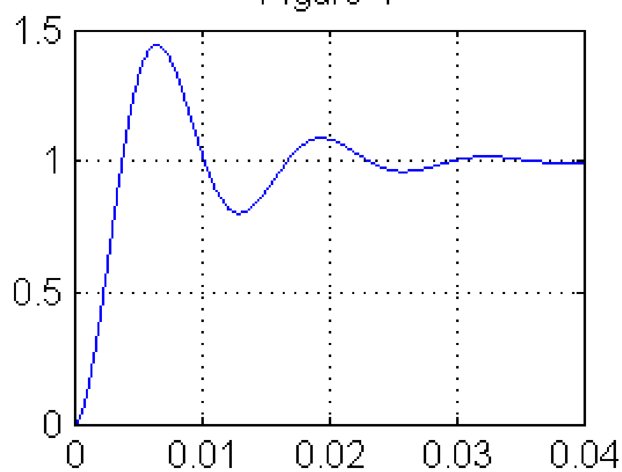


Figure 4



(3A) For each variable listed below, identify the corresponding figure by circling the figure number. *Briefly explain your reasoning in the space below.*

v_C	$\rightarrow$	Figure	1	2	3	4
i_C	$\rightarrow$	Figure	1	2	3	4
i_L	$\rightarrow$	Figure	1	2	3	4

(3B) Estimate the value of R. *A numerical answer with appropriate units is expected.*

(3C) Estimate the value of C. *A numerical answer with appropriate units is expected.*

(3D) Estimate the value of L. *A numerical answer with appropriate units is expected.*

(3E) Estimate the Q of the circuit. *A numerical answer is expected.*

(3F) For each statement below, circle the correct completion. *Briefly explain your reasoning in the space below.*

- If the resistance R is increased, the quality factor Q will $\cdots$

  $\cdots$ increase. $\cdots$ decrease. $\cdots$ remain unchanged.

- If the capacitance C is increased, the quality factor Q will $\cdots$

  $\cdots$ increase. $\cdots$ decrease. $\cdots$ remain unchanged.

- If the inductance L is increased, the quality factor Q will $\cdots$

  $\cdots$ increase. $\cdots$ decrease. $\cdots$ remain unchanged.

(3G) After a long time T, the current source steps down to turn off. For each statement below concerning the subsequent decay of stored energy, circle the correct completion. *Briefly explain your reasoning in the space below.*

- If the resistance R is increased, the time at which the stored energy falls to half its value at time T will $\cdots$

  $\cdots$ increase. $\cdots$ decrease. $\cdots$ remain unchanged.

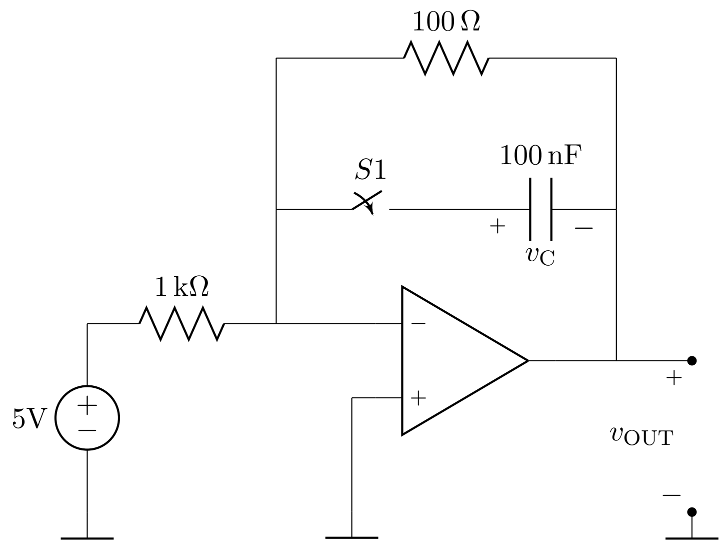
- If the capacitance C is increased, the time at which the stored energy falls to half its value at time T will $\cdots$

  $\cdots$ increase. $\cdots$ decrease. $\cdots$ remain unchanged.

- If the inductance L is increased, the time at which the stored energy falls to half its value at time T will $\cdots$

  $\cdots$ increase. $\cdots$ decrease. $\cdots$ remain unchanged.

Problem 3: Integrator – 25%



Consider the circuit shown above with switch S1 *initially open* and the source is a constant 5V. Assume that the Op-Amp is ideal.

(3A) **6 pts** Find $v_{OUT}(t)$.

$v_{OUT}(t)$=

(3B) **6 pts** Suppose at $t = 0$, switch S1 is moved from an *open to a closed position.* Assume that the capacitor voltage $v_C(0) = 0$ (i.e. there is no initial charge). Find the time constant τ of the system after this event, i.e. for $t > 0$.

$\tau =$

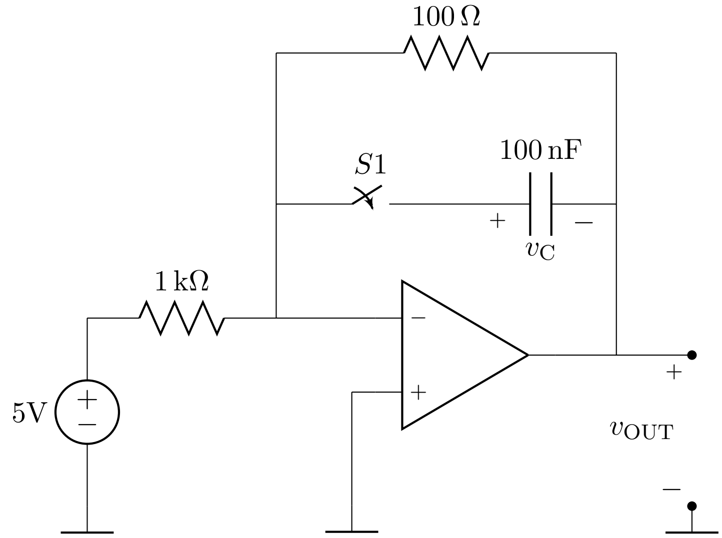
(3C) **7 pts** Solve for $v_{\mathrm{OUT}}(t)$ in the situation described in problem (3B) for $t > 0$. Leave τ as a symbol, rather than substituting your answer from (3B).

For $t > 0$, $v_{\mathrm{OUT}}(t) =$

(3D) **6 pts** At $t = \tau$, assume the switch is switched *back to being open.* Solve for $v_{\text{OUT}}(t)$ for $t > \tau$.

For $t > \tau$, $v_{\text{OUT}}(t) =$

Problem 3: Integrator – 25%



Consider the circuit shown above with switch S1 *initially open* and the source is a constant 5V. Assume that the Op-Amp is ideal.

(3A) **6 pts** Find $v_{OUT}(t)$.

$v_{OUT}(t)$=

(3B) **6 pts** Suppose at $t = 0$, switch S1 is moved from an *open to a closed position.* Assume that the capacitor voltage $v_{\mathrm{C}}(0) = 0$ (i.e. there is no initial charge). Find the time constant τ of the system after this event, i.e. for $t > 0$.

$\tau =$

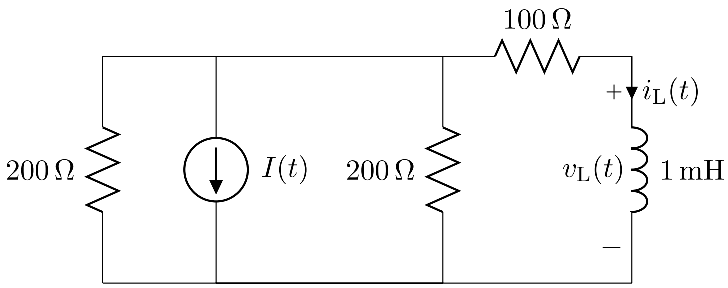
(3C) **7 pts** Solve for $v_{\text{OUT}}(t)$ in the situation described in problem (3B) for $t > 0$. Leave τ as a symbol, rather than substituting your answer from (3B).

For $t > 0$, $v_{\text{OUT}}(t) =$

(3D) **6 pts** At $t = \tau$, assume the switch is switched *back to being open.* Solve for $v_{\text{OUT}}(t)$ for $t > \tau$.

For $t > \tau$, $v_{\text{OUT}}(t) =$

Problem 4: Step Response – 25%



(4A) **6 pts** Assume $I(t) = 10\,\text{mA}$ for $t < 5\,\mu\text{s}$. At $t = 5\,\mu\text{s}$, the strength of this source is suddenly turned to $0\,\text{mA}$. Find the time constant τ of the subsequent response of the system to the transition.

$\tau =$

(4B) **6 pts** Draw $i_{\mathrm{L}}(t)$ vs. t on the grid provided below. Carefully label the limits of the graph and the units on the axes. Neatness counts, so use the spare grids on this page and at the end of your exam if you need them. If you were unable to solve for τ in problem (4A), or are just not confident about your answer, you can use the value $\tau = 10\,\mu\mathrm{s}$.

Spare grid (we suggest you draw a first draft here):

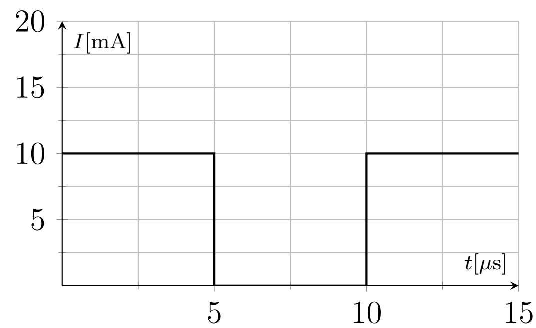
Draw your final answer here:

(4C) **6 pts** Now draw $v_L(t)$ vs. t on the grid provided below. Carefully label the limits of the graph and the units on the axes. Neatness counts, so use the spare grids provided on this page and the separate sheet if you need them. Please use the lower grid on this page for your final answer. If you were unable to solve for τ in problem (4A), or are just not confident about your answer, you can use the value $\tau = 10\,\mu$s.

Spare grid (we suggest you draw a first draft here):

Draw your final answer here:

(4D) **7 pts** At $t = 10\,\mu s$ the strength of the current source is switched back to $10\,mA$ as shown in the graph below. Write an analytic expression for the resulting $i_L(t)$ for $t > 10\mu s$. You may either use your answer for τ given in (4A) or you may use $\tau = 10\,\mu s$ if you were unable to answer (4A).



For $t > 10\mu s$, $i_L(t) =$