Nonlinear Circuit Components and Large-Signal Analysis

Farnaz Niroui, Rm 13-3005B, fniroui@mit.edu
Reference: “Foundations of Analog and Digital Electronics Circuits”, Chapter 4

Nonlinear circuit elements have a nonlinear algebraic relation between their instantaneous terminal current and terminal voltage.

As an example, let's consider a nonlinear device with a square law current-voltage relationship.

\[
 i_D = \begin{cases} 
  K v_D^2 & \text{for } v_D \geq 0 \\
  0 & \text{for } v_D < 0 
\end{cases}
\]

Let's consider a particular case where \( K = 0.1 \).

\[
 i_D = \begin{cases} 
  0.1 v_D^2 & \text{for } v_D \geq 0 \\
  0 & \text{for } v_D < 0 
\end{cases}
\]

Assume this nonlinear device is connected to some arbitrary circuit as shown below. If \( i_B = -1 \, mA \). What is the value of \( v_B \)?

\[
 -i_B = \begin{cases} 
  0.1 v_B^2 & \text{for } v_B \leq 0 \\
  0 & \text{for } v_B > 0 
\end{cases}
\]

Given that \( i_B = -1 \, mA \):

\[
 (-1 \times 10^{-3}) = 0.1 v_B^2 \quad \text{for } v_B \leq 0 \\
 v_B = -0.1 \, V \quad \text{for } v_B \leq 0 
\]

Given that \( V = 2 \, V \), determine \( i \) for the following circuit.

The voltage across each of the nonlinear devices is \( v_D = 2 \, V \).

\[
 i_1 = i_2 = 0.1(2)^2 = 0.4 \, A 
\]
Diode is another common example of a two-terminal nonlinear device. The current-voltage characteristics of a diode is described by the equation below:

\[ i_D = I_S \left( e^{\frac{v_D}{kT}} - 1 \right) = I_S \left( e^{\frac{v_D}{qT_0}} - 1 \right) \]

\( kT/q \): Thermal voltage (based on Boltzman’s constant, temperature and electron charge)

\( I_S \): Based on material and physical design of the diode (for silicon diodes this is typically \(10^{-12}\) A).

There are generally 4 methods for solving nonlinear circuits:

1. Analytical solutions
2. Graphical analysis
3. Piecewise linear analysis
4. Incremental or small signal analysis

Analytical Solutions

Consider the nonlinear resistor circuit shown below.

We can solve this circuit using the node method. Note that the node method which uses the Kirchhoff’s voltage and current laws makes no assumptions about linearity and can be applied to circuits with nonlinear elements. However, the superposition method, the Thévenin method and the Norton method require a linearity assumption.

Write KCL at node 1.

\[ \frac{v_D - E}{R} + i_D = 0 \]
Substitute $i_D = K v_D^2$ for $v_D > 0$.

$$\frac{v_D - E}{R} + K v_D^2 = 0$$
$$R K v_D^2 + v_D - E = 0$$
$$v_D = \frac{-1 + \sqrt{1 + 4 R K E}}{2 R K}$$

$$i_D = K \left[ \frac{-1 + \sqrt{1 + 4 R K E}}{2 R K} \right]^2$$

**Example circuit with one nonlinear device, several sources and resistors**

Consider the circuit shown below with a nonlinear resistor characterized by the following equations.

![Example Circuit](image)

$$i_D = \begin{cases} 
K v_D^2 & \text{for } v_D > 0 \\
0 & \text{for } v_D \leq 0 
\end{cases}$$

The Thévenin method can not be applied to the section of the circuit with the nonlinear device, however, Thévenin equivalent circuit faced by the nonlinear device can be determined (this part of the circuit is linear).

To find the Thévenin equivalent circuit find the Thévenin voltage ($V_{TH}$) and the Thévenin resistance ($R_{TH}$).

1. Find the open circuit voltage (Thévenin voltage ($V_{TH}$)).

$$V_{TH} = V \frac{R_2}{R_1 + R_2} - I_0 R_3$$
2. Find the Thévenin resistance ($R_{TH}$). Set the independent sources to zero.

\[ R_{TH} = (R_1 \parallel R_2) + R_3 \]

3. Connect the nonlinear resistor to the Thévenin equivalent circuit of the linear portion. This can now be solved as discussed for the previous example.

**Example with a diode**

The analytical technique introduced here can be applied to examples with other types of nonlinear devices as well, for example a diode. The following shows a simple circuit with a diode. As shown in the lecture this can be solved using the node method.

Write a KCL at node 1.

\[ \frac{v_D - E}{R} + i_D = 0 \]

\[ i_D = I_S \left( e^{\frac{v_D}{R_{TH}}} - 1 \right) \]

Substitute $i_D$ in the first equation.
\[
\frac{v_D - E}{R} + I_S \left( \frac{v_D}{e^{V_{TH}} - 1} \right) = 0
\]

This does not have a closed form solution, can be solved using trial and error.

**Graphical Analysis**

Graphical analysis can be used to solve circuits that include nonlinear elements. This approach can provide insights about the circuit operation but at the expense of accuracy. Let’s consider the diode example and use a graphical approach for solving it.

As we showed previously, this circuit can be described using the following two equations, both in terms of \(i_D\) and \(v_D\).

\[
i_D = -\frac{v_D - E}{R} = \frac{E}{R} - \frac{v_D}{R}
\]

\[
i_D = I_S \left( \frac{v_D}{e^{V_{TH}} - 1} \right)
\]

To solve these graphically, plot both on the same coordinates (\(i_D\) vs. \(v_D\)) and find the point of intersection.

**Piecewise Linear Analysis**

Circuits containing nonlinear elements can also be solved using the piecewise linear analysis method in which the i-v characteristics is estimated by a succession of linear segments. Then, the calculation for each section can be done using the methods available for linear analysis.

For example, using this method, a diode can be represented based on the ideal diode model shown below.
In the ideal diode model:

Diode ON (short circuit): \( v_D = 0 \) for \( i_D \geq 0 \)
Diode OFF (open circuit): \( i_D = 0 \) for \( v_D < 0 \)

In another model, we use the ideal diode with a voltage offset (voltage source in series with the ideal diode) to more accurately represent the device operation.

The simple diode circuit previously considered can be solved using the piecewise linear method. To do so, replace the diode with its ideal diode model and consider the two resulting linear segments.

\[
\begin{align*}
\text{Short Circuit Segment:} & \quad i_D = \frac{E}{R} \\
\text{Open Circuit Segment:} & \quad i_D = 0 \\
\end{align*}
\]

**An Example of Piecewise Linear Analysis**

The circuit below has a nonlinear element that can be described by the piecewise linear graph shown below.
There are two segments with linear i-v characteristics, each can be described by a resistor as shown below.

\[
v_v = i_D R_1 = I R_1 \quad \text{for } i_D \geq 0 \\
v_v = i_D R_2 = I R_2 \quad \text{for } i_D < 0
\]

Assuming that \( I = 0.002 \cos(\omega t) \), \( R_1 = 100 \, \Omega \) and \( R_2 = 10000 \, \Omega \), find \( v_v \).

\[
v_v = I R_1 = 100I \quad \text{for } I \geq 0 \\
v_v = I R_2 = 10000I \quad \text{for } I < 0
\]