So far, we've seen circuits responding to a step, and circuits responding to a sinusoidal drive. Here, we'll look at both at once. This example shows what happens when a "transient" i.e. a sudden change, hits a sinusoidal steady state system.

Consider the circuit:

\[ i_L(t) + V(t) = 0 \]

Find \( i_L(t) \).

\[ V(t) \quad \text{source} \]
\[ i_L(t) \quad \text{current} \]
\[ V_R \quad \text{voltage} \]
\[ L \quad \text{inductance} \]
\[ R \quad \text{resistance} \]

where \( V(t) \) is a drive signal. It will eventually have various forms, but let's leave it in a general form for now, to help us see the most general solution possible.

Performing KVL around the loop gives us the equation:

\[ V - V_R - V_L = 0 \]
\[ = V - i_L R - L \frac{d}{dt} i_L \]
\[ \Rightarrow \frac{d}{dt} i_L + \frac{i_L}{R} R = \frac{V(t)}{L} \]
\[ \Rightarrow \frac{d}{dt} i_L + \frac{i_L}{\tau} = \frac{V(t)}{L} \]

where \( \tau = \frac{L}{R} \).
This is an inhomogeneous equation, thus must be solved by first finding the homogeneous, then the particular solution, and finally using initial conditions to determine the unknown solution parameters.

Before we proceed, it is worth noting that eq. (1) can be generalized to apply to any variable in circuit (1), e.g. \( v_R \), \( v_L \), etc. by writing

\[
d_x x(t) + \frac{x(t)}{C} = K(t)
\]  

(2)

where \( x \) refers to the variable and \( K(t) \) is a function of the drive. We then seek a solution to the homogeneous form of this equation

\[
d_x x_H(t) + \frac{x_H(t)}{C} = 0
\]

which we know from earlier in the class is

\[ x_H(t) = Ae^{-t/C} \]

Next we seek a particular solution. Assuming such a solution is found, we write

\[ x(t) = x_H(t) + x_p(t) \]

which evidently must satisfy the full form of eq. (2).
To solve for the various coefficients of \( x_h \) and \( x_p \) (e.g., for \( A \)), we use the initial condition.

\[
x(0) = x_i(0) + x_p(0) = A e^0 + x_p(0) = A + x_p(0).
\]

\[\Rightarrow A = x(0) - x_p(0).\]  

We've now taken this about as far as one can go without using specifics of the circuit. Let's go back to determining the current in circuit (1). But now let's specify \( V(t) \).

Suppose first of all that \( V(t) = V_0 u(t) \).

In this case, we might try \( i_{p}(t) = B \) for \( t > 0 \), i.e., assume a constant particular solution. Substituting this form into (1), we find

\[
B/e = V_0/L
\]

\[\Rightarrow B = V_0 e/L\]

From (3) we can then say

\[
A = i_{p}(0) = i_{p}(0) = 0 - V_0 e/L = -V_0/R.
\]

Summing the homogeneous and particular solutions, we find

\[
i_{L}(t) = -\frac{V_0}{R} e^{-t/L} + \frac{V_0}{R}.
\]
It may seem like all we've done is work very hard to recover the standard form:

\[ i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty)) e^{-t/\tau}. \]

Indeed, a quick analysis will show that this approach would give the same result!

So now let's pick a more interesting \( V(t) \).

\[ V(t) = V_0 \cos(wt) u(t). \]

To solve this problem, let's choose a particular solution of the form:

\[ i_L(t) = B e^{jw t} + c.c. \quad (B \text{ is complex}). \]

Substituting this into (1) we find

\[ \left( jw B e^{jw t} + Be^{jw t} / \tau \right) + c.c. = V(t)/L \]

\[ = \frac{V_0}{2L} \left( e^{jw t} + c.c. \right) \quad \text{for } t > 0. \quad (4) \]

At this point, we should observe that

if \[ A e^{jw t} + c.c. = B e^{jw t} + c.c. \]

then \( A = B \). That may seem obvious to some, but if not, just note

\[ A e^{jw t} + c.c. = 2 |A| \cos(wt + \angle A) \]

\[ B e^{jw t} + c.c. = 2 |B| \cos(wt + \angle B) \]

\[ \Rightarrow |A| = |B| \quad \& \quad \angle A = \angle B \quad \Rightarrow A = B \]
Returning to (4) we can now say
\[(\text{j} \omega + \frac{1}{C})B = \frac{V_o}{2L}\]
\[
\Rightarrow B = \frac{V_o}{2L(\text{j} \omega + \frac{1}{C})},
\]
Going back to eq. 3, we can also say
\[
A = 0 - \left( \frac{V_o}{RL(\text{j} \omega + \frac{1}{C}) + \text{c.c.}} \right)
\]
\[
= - \frac{V_o}{L} \text{Re} \left[ \frac{1}{\text{j} \omega + \frac{1}{C}} \right]
\]
\[
= - \frac{V_o}{L \sqrt{\omega^2 + \frac{1}{C^2}}}
\]
\[
\Rightarrow i_L(t) = - \frac{V_o}{L \sqrt{\omega^2 + \frac{1}{C^2}}} e^{-t/C} + \left( \frac{V_o e^{\text{j} \omega t}}{2L(\text{j} \omega + \frac{1}{C})} + \text{c.c.} \right)
\]
\[
= \frac{V_o}{L \sqrt{\omega^2 + \frac{1}{C^2}}} \cos(\omega t + \frac{\text{j} \omega t}{2L(\text{j} \omega + \frac{1}{C})})
\]

Let's try to systematize our understanding of the particular solution. How does the amplitude & phase depend on \(\omega\)?
What does "high frequency" or "low frequency" mean in this context? Notice solution contains a factor \( \frac{1}{(jw + \frac{1}{2})} \).

This factor is sufficient to tell us how the particular responses change with frequency.

Obviously, the frequency changes, so we are not interested in that change.

We want to know how the amplitude and phase change.

Back to the question of "what does low or high frequency mean."

\[
\frac{1}{jw + \frac{1}{2}} = \frac{1}{jw^2 + 1}
\]

From this form it is evident that the effect of \( w \) will be felt when \( w \gg \frac{1}{2} \).
thus we could define "high" frequency when \( w \gg \frac{1}{T} \) and "low" frequency when \( w \ll \frac{1}{T} \).

Looking at \( jw + \frac{1}{T} \) expression graphically, you can see from this drawing that the magnitude and phase of \( jw + \frac{1}{T} \) vary as \( w \) changes.

<table>
<thead>
<tr>
<th>( w \ll \frac{1}{T} )</th>
<th>( w = \frac{1}{T} )</th>
<th>( w \gg \frac{1}{T} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( jw + \frac{1}{T} )</td>
<td>( \approx \frac{1}{T} )</td>
<td>( \approx \frac{1}{2} (j+1) )</td>
</tr>
<tr>
<td>( L(jw + \frac{1}{T}) )</td>
<td>( \approx 0 )</td>
<td>( \approx \frac{\pi}{4} )</td>
</tr>
<tr>
<td>(</td>
<td>jw + \frac{1}{T}</td>
<td>)</td>
</tr>
<tr>
<td>(</td>
<td>\frac{1}{jw + \frac{1}{T}}</td>
<td>)</td>
</tr>
<tr>
<td>( L\left(\frac{1}{jw + \frac{1}{T}}\right) )</td>
<td>( \approx 0 )</td>
<td>( \approx -\frac{\pi}{4} )</td>
</tr>
</tbody>
</table>

Notice that as \( w \) increases, the amplitude drops as \( 1/w \to \) high frequencies have attenuated (reduced) amplitudes.

\[ \text{Diagram of signal attenuation} \]