6.002 Recitation Notes: L-R-C Circuits
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Most resources (including the course text book) treat series and parallel combination of RLC circuits as if they are different circuits. There is nothing wrong with that approach, but I feel it is more natural and helpful to treat these as special cases of a single more general circuit. We will take the single-circuit approach here. I really recommend you also refer to standard texts. This material is intended to supplement, not supplant, the course textbook.

Some Introductory Material

Notation

A* Complex conjugate of A.

ℜ[A] Real part of A.

ℑ[A] Imaginary part of A.

c.c. Complex conjugate of previous term in expression. Thus A + c.c. = A + A*

∂t Derivative with respect to t (technically this is used as a partial derivative, while D is used for the standard derivative, but here we will use it as a convenient notation.

∂tt Second derivative with respect to t (see above).

j Complex constant \(\sqrt{-1}\) (i.e. equivalent to i in math and physics).

Some Useful Identities

\[
A + A^* = 2\Re[A]
\]

\[
e^{ix} + e^{-ix} = 2\cos x
\]

\[
\angle(a + bj) = \tan^{-1}\left(\frac{b}{a}\right)
\]

\[
\cos(x - \frac{\pi}{2}) = \sin(x)
\]

\[
a + bj = |a + bj|e^{\angle(a+bj)}
\]

\[
|a + bj| = \sqrt{a^2 + b^2}
\]
The Problem

The typical LRC circuit consists of a resistor, capacitor, and inductor either in parallel or in a series loop configuration. These two cases are shown in figure 1 below. Typically the problem will provide an initial state for the capacitor (an initial voltage \( v_C(0) \)) or the inductor (an initial current \( i_L(0) \)) or both. You will then be asked to find some current and/or voltage at some later time, or as a function of time.

These two circuits are in fact special cases of a more general circuit, one that arises naturally when one builds an L-C circuit due to the parasitics (undesired circuit elements that arise due to imperfections in the construction and implementation of circuit components in the real world).

Here, we have used \( R_C \) to represent the parasitic parallel resistance of a capacitor,\(^1\) and \( R_L \) to represent the parasitic series resistance of the inductor. It is evident upon inspection that this circuit reduces to a parallel combination of RLC in the limit where \( R_L \rightarrow 0 \), and to a series combination in the limit where \( R_C \rightarrow \infty \).

Of course sources can be added to this problem, and we will discuss that case below. For now, we will assume all source strengths are set to zero and no longer change after \( t > 0 \), but where the capacitor and/or inductor may have a non-zero state at \( t = 0 \). We’ll call the initial inductor state \( \Lambda_0 \) and the initial capacitor state \( Q_0 \).\(^2\)

The problem is to now determine the time evolution for \( t > 0 \) of any of the circuit variables. We will focus our solution on \( v_C, i_L \), but the exact approach would work for any other variable. Indeed, the

\(^1\) Physically, this resistance represents the small leakage of current through the capacitor dielectric.

\(^2\) \( Q_0 \) on a capacitor will result in a voltage \( V_c = Q_0 / C \). Similarly \( \Lambda_0 \) will result in a current \( I_c = \Lambda_0 / L \).
advantage of taking this more general approach is that we cast the broadest possible net in solving the problem.

**Full Derivation**

We will start with a full derivation of the solution to the most general form of the problem. We will then provide a couple examples in which specific circuits are analyzed.

**Step 1: Deriving the Differential Equation**

From the constitutive relations for a capacitor and an inductor, we can write

\[ i_C = C \frac{dv_C}{dt}, \quad \text{and} \quad v_L = L \frac{di_L}{dt}. \tag{1} \]

We can then use KVL around the \( L-R-L-C \) loop to derive the equation:

\[ v_C = v_L + i_L R_L. \tag{2} \]

We can also use KCL at either node to state:

\[ i_L = -i_C - v_C G_C \tag{3} \]

where we have defined \( G_C \equiv 1/R_C \). Substituting (1) into (2) and (3) we get:

\[ v_C = L \frac{di_L}{dt} + i_L R_L \tag{4a} \]

\[ i_L = -C \frac{dv_C}{dt} - v_C G_C. \tag{4b} \]

These equations reduce to the same coupled first-order differential equations as arise in an L-C circuit when \( R_L \to 0 \) and \( G_C \to \infty \).

In this format, the solution is quite computable by numerical methods, and in practice this is a convenient way to approach the problem. However, such an approach does not provide the necessary intuition, so we will take the step of reducing these equations to an equation of a single variable.

To derive an equation in terms of only \( i_L \), we will now substitute (4a) into (4b):

\[ -i_L = LC \partial_t i_L + R_L C \partial_t v_C + L G_C \partial_t i_L + G_C R_L i_L \]

\[ \Rightarrow 0 = \partial_t i_L + \left( \frac{R_L}{L} + \frac{G_C}{C} \right) \partial_t v_C + \frac{1 + R_L G_C}{LC} i_L. \tag{5} \]

Remarkably, if we do the opposite and substitute (4b) into (4a) the form of the equation doesn’t change, just the variable, so

\[ -v_C = LC \partial_t v_C + R_L C \partial_t v_C + L G_C \partial_t v_C + G_C R_L v_C \]

\[ \Rightarrow 0 = \partial_t v_C + \left( \frac{R_L}{L} + \frac{G_C}{C} \right) \partial_t v_C + \frac{1 + R_L G_C}{LC} v_C. \tag{7} \]

It turns out that, due to a deep duality between electric and magnetic fields that originates in special relativity, conductance is the more natural variable to use when dealing with inductor parasitics.

Here we introduce the notation \( \partial_t \) for \( d/dt \), and \( \partial_{tt} \) for \( d^2/dt^2 \) simply because it is easier to use when doing algebra.
Even more remarkably, it turns out that any circuit variable we choose (even \( i_R \) or \( v_R \)) will have exactly the same form. To write this most generally, we will use \( x \) to represent \( v_C \) or \( i_L \) or any variable we may be interested in, and so we can write:

\[
\partial_t x + \left( \frac{R_L}{L} + \frac{G_C}{C} \right) \partial_t x + \frac{1 + R_L G_C}{LC} x = 0.
\] (9)

Isn’t linearity a miraculous thing?

As a bit of an aside, it is useful to note that this more complicated form reduces to the more simple series and parallel combination forms available in most textbooks by setting \( R_L \to 0 \) (parallel) or \( G_C \to 0 \) (series).

**Step 2: Identifying Some Special Constants**

Equation (9) is an important form for us, but the coefficients are a bit cumbersome. Anticipating some of the features of the solution (sorry... it will be clear why we do this soon), we rewrite it in an even more general form as:

\[
\partial_t x + 2\alpha \partial_t x + \omega_0^2 v_C = 0.
\] (10)

where

\[
\alpha \equiv \frac{R_L}{2L} + \frac{G_C}{2C}.
\] (11)

\[
\omega_0^2 = \frac{1 + R_L G_C}{LC}.
\] (12)

Again, notice that (12) reduces to the usual form when \( R_L \) or \( G_C \) (or both) go to zero. Also, although \( \alpha \) has two terms here, in situations we will consider in this course, either \( R_L \) or \( G_C \) will always be zero, so it will generally only have one term.

**Step 3: Finding a Solution**

Let’s try a solution of the form \( Ae^{st} \).\(^5\) Substituting this into (10), we find

\[
s^2 Ae^{st} + 2\alpha s Ae^{st} + \omega_0^2 Ae^{st} = 0
\] (13)

\[
\Rightarrow s^2 + 2\alpha s + \omega_0^2 = 0.
\] (14)

which is known as the “characteristic equation” of this system.

The solutions to this equation by the quadratic formula will be

\[
-\frac{b}{2\alpha} \pm \frac{\sqrt{b^2 - 4ac}}{2\alpha}
\] (15)

\(^5\) I know this seems unsatisfying... but this is literally the only differential equation—and the only solution—you’ll ever see in circuit theory. So can we just accept it and move on?
where \( a = 1, b = 2a, \) and \( c = \omega_0^2, \) which is then

\[
s = -\frac{2a}{2} \pm \frac{\sqrt{4a^2 - 4\omega_0^2}}{2}
\]

\[
s = -a \pm \sqrt{a^2 - \omega_0^2}.
\]

This trial solution naturally gives us three cases, based on the sign of \( a^2 - \omega_0^2. \) If \( \omega_0 < a, \) the situation is termed “underdamped” for reasons we shall see in a moment. If \( \omega_0 = a, \) the situation is termed “critically damped” and is of little interest to us. And finally, if \( \omega_0 > a, \) the situation is termed underdamped, again, for reasons we shall see in a moment. We will primarily concern ourselves with the underdamped case, because it has many applications in the real world, but overdamped circuits are also useful in many situations.

**The Underdamped Case**

In the underdamped case, where \( a < \omega_0, \) the expression \( a^2 - \omega_0^2 \) from (17) will be negative. In that case, we can rewrite (17) as:

\[
s_\pm = -a \pm j\omega_d
\]

where we use \( j \) as the imaginary constant instead of \( i \) to avoid confusion with current,\(^6\) and we define a new frequency parameter

\[
\omega_d \equiv \sqrt{\omega_0^2 - a^2}.
\]

When the damping term \( a \) is small relative to \( \omega_0, \) \( \omega_d \approx \omega_0. \)

The fact that there are two roots for the characteristic equation in this case suggests that two valid solutions will exist. Because it is a linear equation, superposition of these two solutions should also be a valid solution, thus generally, the solution should be of the form:\(^7\)

\[
x = A_+ e^{s_+t} + A_- e^{s_-t}
\]

\[
= A_+ e^{-at} e^{j\omega_dt} + A_- e^{-at} e^{-j\omega_dt}
\]

\[
= e^{-at} (A_+ e^{j\omega_dt} + A_- e^{-j\omega_dt}).
\]

Next, we have to determine the values of \( A_+ \) and \( A_- \). Before we go there, let’s talk about the (literal) complexity of our proposed solution. You have every right at this point to be very concerned that our proposed solution appears to be complex! After all, \( e^{j\omega_dt} \) is certainly complex... But we know that “real” voltages and currents cannot be complex. So what gives? Well, the key is that \( A_+ \) and \( A_- \) must be complex conjugates of each other... so we can write \( A_+ \equiv A \) and then \( A_- = A^*. \) As a result, the term \( A e^{j\omega_dt} \) is the

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\(^6\) Using \( j \) instead of \( i \) is standard practice in electrical engineering, to avoid confusion with the current symbol. But why is current denoted \( i \) instead of \( c \) (or \( j, \) even)? Because it is derived from the French term “intensité,” meaning “intensity” which originated before people understood that current represented a flow of particles.

\(^7\) The case we are working here is for a homogeneous equation, thus this will apply for the homogeneous solution. Circuits with voltage sources will be slightly different, as we will discuss below.
complex conjugate of $A^*e^{-j\omega t}$ and their sum will be real, so $x$ will be real. We can finally then write:
\[
x(t) = e^{-at} \left( A e^{j\omega t} + A^* e^{-j\omega t} \right) = e^{-at} \left( A e^{j\omega t} + \text{c.c.} \right)
\]
where we introduce the notation c.c. to represent the complex conjugate of the first term in the expression.\(^8\)

**Using Initial Conditions**

Now, back to figuring out $A_+$ and $A_-$ (or just $A$ now that we know they're just complex conjugates). There are two unknowns (the real and imaginary parts of $A$), so we are going to need two pieces of information about the circuit to solve for them. In principle we could use the values of $x$ at any two points in time, but the $t = \infty$ case does us no good because $e^{-at}$ factor is zero there, so we lose any useful information. Luckily, we have been given two facts that we can use as initial conditions.

Although we have two initial conditions, there is only one variable $x$ so there is only one $x(0)$. Luckily, we can use $\partial_t x(0)$ as one of our conditions. But “we don’t know $\partial_t x(0)$,” do I hear you cry? Not so! We may not have been told it explicitly, but the current can be used to calculate the derivative of the voltage and vice versa. This can be figured out from (4a) and (4b). Rearranging these expressions, we can write:

\[
\begin{align*}
\partial_t i_L &= \frac{v_C}{L} - \frac{R_L}{L} i_L \\
\partial_t v_C &= -\frac{i_L}{C} - \frac{G_C}{C} v_C
\end{align*}
\]

Because we know $i_L(0) = \Lambda_o / L$ and $v_C(0) = Q_o / C$ (remember this was part of the initial setup of the problem?), we can substitute for these values above and find:

\[
\begin{align*}
\frac{di_L(0)}{dt} &= \frac{Q_o}{LC} - \frac{R_L \Lambda_o}{L^2} \\
\frac{dv_C(0)}{dt} &= -\frac{\Lambda_o}{LC} + \frac{G_C Q_o}{C^2}.
\end{align*}
\]

This is more complex than typical... normally, either $\Lambda_o$ or $Q_o$ will be zero, and these conditions will simplify greatly as a result.

So we now know $x(0)$ and $\partial_t x(0)$. We can find the two equations for these expressions be evaluating (24) at $t = 0$, and by taking its

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\(^8\) Complex conjugation is just the process of replacing all the $j$ symbols with $-j$ symbols, i.e. multiplying the imaginary part of the number by $-1$. 
derivative, and then evaluating that at $t = 0$.

\[ x(0) = A + c.c \]  
\[ \left. \frac{\partial_t x(t)}{t=0} \right| = \left. \left( -ae^{-at} \left( Ae^{j\omega_d t} + c.c. \right) + e^{-at} \left( j\omega_d Ae^{j\omega_d t} + c.c. \right) \right) \right|_{t=0} \]
\[ = -(\alpha + j\omega_d)A + c.c. \]  
\[ (29), (30), (31) \]

\[ (29), (31) \] are just two equations and have two unknowns (remember, $x(0)$ and $\partial_t x(0)$ are now both known, set by the initial conditions on current and voltage given in the problem), and so can be solved by using standard linear algebra methods. Making the observation that the coefficient of $A$ in (31) is just $s_-$ we can write:

\[
\begin{pmatrix} x(0) \\ \partial_t x(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ s_- & s_+ \end{pmatrix} \begin{pmatrix} A \\ A^* \end{pmatrix}
\]

which can be inverted to give

\[
\begin{pmatrix} A \\ A^* \end{pmatrix} = \frac{1}{s_+ - s_-} \begin{pmatrix} s_+ & -1 \\ -s_- & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ \partial_t x(0) \end{pmatrix}
\]

\[ (32), (33) \]

The Final Answer

Now that we know $A$, we can back-substitute into (24) to write out the full expression for $x(t)$:

\[ x(t) = e^{-at} \left( \left( \frac{-\alpha + j\omega_d}{2j\omega_d} x(0) - \frac{1}{j\omega_d} \partial_t x(0) \right) e^{j\omega_d t} + c.c. \right) \]  
\[ (35) \]

While this is indeed a daunting equation... for simpler cases, it is immediately reducible. For example, in many cases either $x(0)$ or $\partial_t x(0)$ will equal zero.

Conveniently, this form can always be reduced to a more intuitive form as follows:

\[ x(t) = Xe^{-at} \cos (\omega_d t + \phi_0). \]  
\[ (36) \]

where $^9$

\[ X = \left| \frac{-\alpha + j\omega_d}{j\omega_d} x(0) - \frac{1}{j\omega_d} \partial_t x(0) \right| \]  
\[ (37) \]

which is real, and

\[ \phi_0 = \angle \left( \frac{-\alpha + j\omega_d}{j\omega_d} x(0) - \frac{1}{j\omega_d} \partial_t x(0) \right), \]  
\[ (38) \]

$^9$ We have used the relation $Ae^{\angle B} + c.c. = 2|A| \cos (B + \angle A)$, where the $\angle$ symbols represents the polar angle of its operand.
which is of course also real. This form strongly suggests that these solutions will always have the form of a decaying oscillation. The cosine term enforces oscillation with radial frequency $\omega_d$ in units of radians per time ($\omega_d/2\pi$ in cycles per unit time). The exponential term multiplies the cosine and causes it to decay with a time constant of $\tau = 1/\alpha$.

**Examples**

Let’s apply our solution to two examples. Suppose that $G_C = 0$, i.e. the resistor in parallel with the capacitor has infinite resistance. In that case, it is just like an open circuit, and we can ignore it.

Suppose the inductor starts out un-fluxed (i.e. $i_L(0) = 0$), and the capacitor starts out with some initial voltage across it $V_o$. Find the current through the inductor and voltage across the capacitor as a function of time in this circuit.

We’ll deal with finding the current in the inductor first.

First, let’s set up the differential equation for the problem. We will use our definitions of $\alpha$, $\omega_o$, and $\omega_d$, substituting in $G_C = 0$ to find:

$$\alpha = \frac{R_L}{2L}, \quad \omega_o = \frac{1}{\sqrt{LC}}, \quad \omega_d = \sqrt{\frac{1}{LC} - \frac{R_L^2}{4L^2}} \quad (39)$$

We could now write out our differential equation... but we won’t even bother! We just recognize we are trying to solve for $i_L$, so we should set $x \equiv i_L$ in our solution above and jump straight to the solution as shown in (10).

To determine the initial conditions, we observe directly that we were told that $i_L(0) = 0$, which is our $x(0)$. We only need to determine $\partial_t x(0)$ which we can from our equation (27) above, recognizing that if $i_L(0) = 0$, then $\lambda_o = 0$ and if $v_C(0) = V_o$, $Q_o = CV_o$ so $\partial_t i_L(0) = V_o/L$, which is our $\partial_t x(0)$ condition.

With these two initial conditions, we can use our solution above to
write:

\[ i_L(t) = e^{-\alpha t} \left( -\frac{V_o}{2j\omega_d L} e^{j\omega_d t} + \text{c.c.} \right) \]  \hspace{1cm} (40)

\[ = \frac{V_o}{\omega_d L} e^{-\alpha t} \cos (\omega_d t + \pi/2) \]  \hspace{1cm} (41)

\[ = -\frac{V_o}{\omega_d L} e^{-\alpha t} \sin (\omega_d t) \]  \hspace{1cm} (42)

where we have used our expressions (37) and (38). Additionally, in the second step we used the fact that \( \angle(-1/j) = \pi/2 \), and in the third step we used the fact that \( \cos(\theta + \pi/2) = -\sin(\theta) \).

We were also interested in \( v_C(t) \) for this problem. We know that \( \alpha \), \( \omega_o \) and \( \omega_d \) are unchanged, so we can skip straight to the initial conditions. In this case we observe that \( v_C(0) = V_o \) and because \( i_L(0) = 0 \), the \( \partial_i x(t) \) term will be zero. With these two initial conditions, we can use our solution above in a single step as:

\[ v_C(t) = e^{-\alpha t} \left( -\frac{\alpha + j\omega_d}{2j\omega_d} V_o e^{j\omega_d t} + \text{c.c.} \right) \]  \hspace{1cm} (43)

\[ = V_o e^{-\alpha t} \left( -\frac{\alpha + j\omega_d}{2j\omega_d} e^{j\omega_d t} + \text{c.c.} \right) \]  \hspace{1cm} (44)

Let’s pause for a second to point out that the coefficient of the \( e^{j\omega_d t} \) term can be written in polar notation as

\[ \frac{-\alpha + j\omega_d}{2j\omega_d} = e^{j\phi_o} \sqrt{1 + \frac{\alpha^2}{\omega_d^2}} \]  \hspace{1cm} (45)

where we have defined \( \phi_o \equiv \angle(-\alpha + j\omega_d) - \angle j \). We can use the fact that \( \angle j = \pi/2 \) and in general \( \angle(a + jb) = \tan^{-1}(b/a) \), where one has to take care to get the correct quadrant of the arctan function, to write:

\[ v_C(t) = V_o e^{-\alpha t} \left( e^{j(\omega_d t + \phi_o)} + \text{c.c.} \right) \]  \hspace{1cm} (47)

\[ = V_o e^{-\alpha t} \sqrt{1 + \frac{\alpha^2}{\omega_d^2}} \cos (\omega_d t + \phi_o), \]  \hspace{1cm} (48)

where \( \phi_o = \angle(-\alpha + j\omega_d) - \angle j = \tan^{-1}(-\omega_d/\alpha) - \pi/2 \) and we know that we are in the correct quadrant because \( \alpha \) and \( \omega_d \) are both real and positive.

Finally, it is most instructive to study the two solutions together and graphically.
Figure 2: Characteristic decaying oscillation observed in RLC circuits. Time-domain comparison of $i_L$ (red) and $v_C$ (blue), normalized to have the same amplitude at $t = 0$. Current is needed to charge and discharge the capacitor, thus it leads the capacitor voltage. Period of oscillation (time between zero crossings) is $T = 2\pi/\omega_d$.

Figure 3: Parametric plot of current in inductor vs. voltage across capacitor, showing oscillation. Current is multiplied by “characteristic impedance” $\omega_d L$ to give it units of volts, and to permit plotting on the same scale as voltage, otherwise plots would appear elliptical. System spirals in with time.
What’s Next?

Derived parameters like $\omega_c$, $\alpha$, and $\omega_d$ are tremendously helpful when trying to determine quickly how a circuit behaves. But there are even more parameters that we haven’t discussed here that can further help with interpretation. Discussion of how energy moves back and forth between circuits, how it is dissipated, and how overdamped cases should be treated are all interesting and worthwhile areas to look into. Furthermore, circuits that at first glance don’t appear to fall into the simple structure shown can arise, but with some effort can be mapped onto this structure. Ultimately, using the frequency domain rather than the time domain to analyze circuit behavior provides even more powerful methods.

Glossary and Definitions

**Characteristic Equation:** Polynomial equation used to determine the eigenvalues of a matrix. In this case, refers to the polynomial form equation that results from substitution of a trial solution.

**Complex Conjugate:** Takes the imaginary part of a complex number and multiplies it by -1. The real part of the number is unchanged. Thus $(a + bj)^* = (a - bj)$ where $a$ and $b$ are real numbers, and $j$ is the imaginary constant $j^2 = -1$.

**Critically Damped:** An approximate definition is the situation in which damping rate in a system matches the rate of oscillation or equivalently the decay time constant matches the oscillation period. This regime is of little interest to us here, but of value in advanced topics, particularly in mechanics.

**Duality:** Inductors and capacitors are often referred to as “dual” circuit elements. Duality means that the role of current and voltage are reversed. Thus we could also say that resistance and conductance are dual variables. It turns out that, geometrically, a node and a loop in a circuit are also dual topological elements. Duality is a deep concept in circuits, and covered in detail in advanced classes.

**Initial Conditions:** Values of circuit variables at the initiation of a region in which a differential equation is applied.

**Natural Frequency:** The natural frequency $\omega_c$ is the frequency at which a perfect L-C oscillator would resonate in the absence of a driving source.
**Overdamped:** Situation in which damping rate in a system is large relative to the rate of oscillation or equivalently the decay time constant is short relative to the oscillation period. Rapid non-oscillatory decay is characteristic of this regime.

**Parasitics:** Undesired circuit elements that arise due to imperfections in the construction and implementation of circuit components in the real world. For example, series resistance is unavoidable in inductors made of normal metals (not superconductors), and a parallel resistor is unavoidable in capacitors.

**Radial Frequency:** Frequency expressed in units per unit time instead of the more conventional cycles per unit time. There are $2\pi$ radians in a cycle, thus radial frequency $\omega$ is related to conventional frequency $f$ through the relation $\omega = 2\pi f$.

**Underdamped:** Situation in which damping rate in a system is small relative to the rate of oscillation or equivalently the decay time constant is long relative to the oscillation period. Decaying oscillations are characteristic of this regime.