Potential vs. Voltage

In Electrical Engineering we often use the terms “potential” and “voltage” interchangeably, and it is easy and tempting to do so. But when getting started, that can be confusing. Voltage refers to a potential difference between points. I will use the term “potential” or “node voltage” to refer to the voltage difference between a node and a reference ground. I will use “branch voltage” or simply “voltage” to refer to a difference in potential across a single element.¹

This drawing shows a generic circuit element with the nodes labeled with their potential values ($e_A$ and $e_B$) and the branch voltage (i.e. $v = e_A - e_B$) labeled.

As you move from the ‘+’ labeled terminal of a circuit element to the ‘−’ terminal, if $v > 0$ the ‘−’ terminal potential will have a lower value than the ‘+’ terminal. Of course if $v > 0$, the opposite will be the case.

Very often, the voltage is unknown, so the node potentials are similarly unknown. However, there is one particularly simple case: when the circuit element is a voltage source with strength $V_S$, as shown below.

$$e_A = e_B + V_S \quad \text{or} \quad e_A = e_B + V_S$$

This observation may seem simple or obvious at the moment, but it will be important when we talk about supernodes later, so I’m emphasizing it here.²

The constituitive relation for the resistor (Ohm’s law), can be represented just as easily with potentials as with voltage:

$$\frac{1}{R}$$

The constituitive relation is normally written $v = iR$ or $i = Gv$ where $G = 1/R$ is the conductance. However, now this expression

¹ This situation is slightly confusing because both measures use the same units of volts—unfortunately that can’t be avoided.

² Defining a unique potential value at each node ensures that Kirchhoff’s voltage law is satisfied. You can prove this with a bit of math, if you’re feeling motivated.
can be written equivalently as \( i = (e_A - e_B)G \). This may seem like a minor observation, but we will use it extensively as we apply the node method, so it is important to be comfortable with it.

What is a Node?

There are a number of ways to draw a node, and a number of ways to talk about it. From a mathematical perspective (in graph theory), a node is strictly a point that connects to lines (branches). Intersections of branches should always be marked by a dot in circuits, as shown below.

![Node representation](image)

However, from the perspective of potentials, a single node potential can be shared across many physical nodes (i.e. branch intersections), if those nodes are all shorted together, i.e. connected by zero-resistance wires. When considering potential, we will loosely refer to such an assembly as a ‘node’ even though of course it is not quite technically a single node.\(^3\)

The dashed line below outlines the collection of two nodes, in this case, that share a single node potential. We will call this a node, even though from a mathematical/graph theory perspective, it is actually two nodes.\(^4\)

![Node collection](image)

Example

To test your comfort level with a node, consider the circuit diagram shown below. Try to solve for the effective resistance between nodes \( A \) and \( B, R_{AB} \).

![Circuit diagram](image)

\(^3\) You may remember from your physics classes that we are free to choose any node we wish to be our reference (‘ground’). We will specify this reference node with the following symbol:

![Reference node symbol](image)

where the open circle will connect to the reference node.

\(^4\) It is common in complicated circuit schematic diagrams to use labels to refer to nodes that are implicitly connected by a short circuit. This practice reduces the number of lines required to draw the circuit, and reduces clutter.
Supernode

The term “supernode” is used when referring to a subcircuit that we will treat as a single node from the point of view of Kirchhoff’s current law.

Although we think of Kirchhoff’s current law as applying only at a true (mathematical) node, it can be used to show that the sum of the currents into (or out of) any simply connected subcircuit must be zero. This is because the underlying physical principle (constant charge density in the circuit) applies to whatever volume of space we select. This theorem can be derived from Kirchhoff’s current law—it’s a worthwhile exercise to try, if you feel so inclined.

For this system, \( i_3 - i_1 + i_2 = 0 \), regardless of what is contained in the circle. Applying this same concept to a circuit is shown below.

For the indicated subcircuit boundary, we can write the subcircuit variant of Kirchhoff’s law as \( i_1 + i_2 + i_4 - i_5 = 0 \). Notice, that we \( i_3 \) does not appear in this form, as it does not cross one of the region boundaries.

We call the subcircuit contained in the indicated boundary a ‘supernode’ and it will come in useful in node analysis when dealing with voltage sources that are not connected to a ground.
Node Method

Armed with an understanding of what a node is and what a node potential means, Kirchhoff’s current law, and the concept of a supernode, we will be able to solve some very challenging circuits quite quickly.

The node method is not fundamentally different than any other way of analyzing circuit. The inputs are the same: the network topology, Kirchhoff’s laws, and the constitutive relations of the circuit elements. A computer could simply accept these inputs and perform the analysis. However, the node method is a process for applying these inputs that minimizes the complexity of the algebraic expression that results. It is easier to construct and to solve than other approaches.

More importantly, the node method introduces a way to think about circuits that will permit you to accumulate the right kind of intuition and will eventually permit you to work with circuits quickly and creatively. Thus, it is key to master this process.

We will introduce the node method by way of the following example.\(^5\)

\[\text{Diagram with nodes and resistors}\]

Notice that $R_2$ and $R_4$ have leads that cross but do not connect. You can tell that they do not connect because there is no solid dot drawn at their intersection. \(^6\)

The node method has 6 major steps.

1. **Identify your nodes**: To use the node method we have to know what our nodes are... but as we discussed at the start of these notes, that can be tricky. It is made even more tricky because we might need to use supernodes to simplify the analysis, and these

\(^\text{5}I\text{ have deliberately drawn this diagram to not be in standard form (unlike most diagrams you’ll find in textbooks), because at this stage in the class practice working with the topology of circuits is still very beneficial.}\)

\(^\text{6}Professional\text{ circuit designers and design software avoid ever having four wires connect at the same point, and will insert a small offset between two of the wires to avoid such situations, as shown below.}\)
must also be identified at this step. Luckily, in this first example, supernodes won’t be necessary. We start our analysis by looking at the circuit and figuring out what the nodes are. At first, you may find it helpful to actually highlight them, as shown below.

At this point, we can make a simplification of the circuit if we want to. It won’t change the math much, but it will simplify the way we think about the problem a bit, so I would say it is worth it. To achieve the simplification, look at the circuit above and think for yourself if any of these resistors are in series or parallel, and thus can be reduced?

It might not be obvious at first, but $R_2$ and $R_3$ connect between the same nodes (green and orange/brown) are thus are in parallel. We can thus reduce them to a single resistor of value $R_2 // R_3$, and will do so in our subsequent diagrams.

2. **Choosing the Ground Node:** In this next step, we want to select whichever single node connects to the most voltage sources to be our ground node. This will only be the bottom node if the circuit is drawn in standard form, which will not always be the case. Also, if there is a tie, we can just pick one of the tied nodes at random.

In the example shown, the top-left (blue) node is connected to both voltage sources, and is thus the best choice for the ground. We label it accordingly, using our conventional reference ground symbol, as shown below.
3. Identify any floating voltage sources and label the supernodes:
If any of the voltage sources are not connected to ground, they are “floating” and will be used to form supernodes. We’ll explain this step later, for now you can just ignore it, as in this example there are no floating voltage source.

4. Label the node potentials: This process has two or three sub-steps, the details of which depend on the circuit topology.

(a) Label Known Potentials: Some of the node potentials are immediately determined given the ground and the strength of the voltage supplies connected to ground. For each of these nodes, label them with the appropriate voltage (being careful of the sign of the source).

(b) Label unknown node potentials: For each remaining node, label it with a variable name. By convention we use $e_1, e_2, ...$ to label our unknown (variable) node potentials.

5. Write out Kirchhoff’s laws for each node (or supernode—to be discussed in a later example) that includes all the branches inputting current to that node.
Let’s do that now for the node associated with potential $e_2$. We are going to skip one step in that process that in the past we’ve included—we are not going to define current variables at all. Instead, we are just going to use the node potentials and ohm’s law to write down the current in a single step.

For example (for the node associated with $e_1$), consider the current through $R_1$ entering the node of interest. The voltage across the resistor is $V_1 - e_1$ so the current through it, from Ohm’s law, is just $(V_1 - e_1)/R_1$. If you remember the concept of conductance, we can write this equivalently as $(V_1 - e_1)G_1$ where $G_1(V_1 - e_1)$ where $G_1 = 1/R_1$. I prefer working with conductance when writing down current because it avoids fractions, which I’ve never really liked. For the rest of the problem, we will use conductance instead of resistance, so $G_2 = 1/R_2$, $G_3 = 1/R_3$, $G_4 = 1/R_4$, $G_5 = 1/R_5$ and, conveniently, the parallel combination conductance $1/(R_3//R_2) = G_2 + G_3$.

Getting back to $e_2$, we can sum the currents into that node, starting with the current through $R_5$.

$$(-V_2 - e_2)G_5 + (e_1 - e_2)(G_2 + G_3) = 0.$$  

Now we can do the exercise for $e_1$, starting with $R_1$, then dealing with each branch going around the node clockwise.

$$(V_1 - e_1)G_1 - I_0 + (0 - e_1)G_4 + (e_2 - e_1)(G_2 + G_3) = 0.$$  

6. Solve for Node Potentials: Looking carefully at these two equations, you’ll notice that only two unknown variables exist... the rest are element parameters, i.e. resistor values or source strengths, and are thus known. **We have thus reduced a circuit with a large number of unknown variables (every current and voltage across 7 elements, or 14 unknowns!) to two unknowns, simply by approaching the problem strategically.**

Any number of computer software tools can now be used to solve this problem. My preferred tool of choice is Mathematica, so I will illustrate that method here, but any method you like will work. A number of pocket calculators are even able that can solve this problem.
6.002 Recitation Notes: The Node Method Including Supernodes

\[ \begin{align*}
\text{eq1} &= (-V_2 - e_2) G_5 + (e_1 - e_2) (G_2 + G_3) = 0; \\
\text{eq2} &= (V_1 - e_1) G_1 - I_0 + (0 - e_1) G_4 + (e_2 - e_1) (G_2 + G_3) = 0; \\
\text{soln} &= \text{Solve}\{\text{eq1, eq2}, \{e_1, e_2\}\}; \\
\text{FullSimplify}\[\text{soln} \/. \{G_1 \rightarrow 1/R_1, G_2 \rightarrow 1/R_2, G_3 \rightarrow 1/R_3, G_4 \rightarrow 1/R_4, G_5 \rightarrow 1/R_5\}]
\end{align*} \]

Out[4]= \[
\left\{ \begin{array}{l}
e_1 \rightarrow -\frac{R_4 (R_3 R_5 + R_2 (R_3 + R_5))}{R_3 R_4 R_5 + R_2 R_4 (R_3 + R_5) + R_1 R_3 (R_4 + R_5) + R_1 R_2 (R_3 + R_4 + R_5)} (I_0 R_1 - V_1) + R_1 (R_2 + R_3) R_4 V_2 \\
e_2 \rightarrow -\frac{(R_2 + R_3) R_4 R_5 (I_0 R_1 - V_1) + (R_1 R_2 R_3 + R_2 R_3 R_4 + R_1 R_2 R_3 R_4) V_2}{R_3 R_4 R_5 + R_2 R_4 (R_3 + R_5) + R_1 R_3 (R_4 + R_5) + R_1 R_2 (R_3 + R_4 + R_5)} \end{array} \right. 
\]

The problem of course can also be solved by hand using standard linear-algebraic methods.

7. **Use Node Potentials**: The node potentials are typically not an end in themselves. We can only measure voltage differences or currents, not abstract node potentials. Thus problems usually ask us to determine one of these measureable quantities. However, they are readily determined based on the calculated node potentials. In this case, we are asked to find \(i_5 = (-V_2 - e_2)/R_5\).

\[
i_5 = \frac{(R_2 + R_3)(I_0 R_1 R_4 - R_1 V_2 - R_4 (V_1 + V_2))}{R_1 R_2 (R_3 + R_4 + R_5) + R_1 R_3 (R_4 + R_5) + R_2 R_4 (R_3 + R_5) + R_3 R_4 R_5}.
\]

**Node Method with Floating Voltage Sources**

Some circuits do not yield to the standard node method. Because the node method requires us to sum the currents, expressed in terms of node potentials, it can only be trivially applied in the case of current sources or resistors connecting to the node. When a voltage source is connected to a node with unknown potential, another method must be used. Below is an example of such a case. Suppose, in this case, we wanted to find the indicated current \(i_3\) through \(R_3\).

[Diagram of the circuit with \(I_0\), \(R_1\), \(R_2\), \(V_1\), \(R_3\), \(R_4\), and \(V_2\).]

Looking at this circuit and working through the node method above, we can select any of the nodes connected to our voltage...
sources as ground (because no node has more voltage sources connected to it than any other). Let’s choose the bottom node to be the ground. The supernode will then include the voltage source $V_1$ and we will sum the current from all the branches that connect to it, as shown here encircled in red.

Now, when labeling the nodes, we will include only one node variable for the supernode (say $e_1$). The other side of the voltage source will be labeled as a sum set by the source strength (in our case $e_1 + V_1$).

We can then write down the resulting node equations. Starting with the node associated with $e_2$, and assuming $G_n \equiv 1/R_n,$

$$I_o - e_2 G_1 + (e_1 + V_1 - e_2) G_2 = 0.$$  

For the supernode, we need to look at all the branches that put current into it. These include $R_2$, $R_3$, and $R_4$. Working through each of these in turn (but ignoring everything inside the supernode), we find

$$(e_2 - (e_1 + V_1)) G_2 - e_1 G_3 + (-V_2 - e_1) G_4 = 0.$$  

Again, we find ourselves with two equations and two unknowns, which we solve using Mathematica (or your calculator, or whatever solver you prefer—you can even do it by hand if you enjoy that kind of thing). If you wish to do it yourself you can check your answer against mine, below:

$$e_1 = \frac{I_o R_1 R_2 R_4 - R_3 (V_2 (R_1 + R_2) + R_4 V_1)}{R_4 (R_1 + R_3) + R_3 (R_1 + R_2)},$$

$$e_2 = \frac{R_1 (I_o R_4 (R_2 + R_3) + L R_2 R_3 + V_1 (R_3 + R_4) - R_3 V_2)}{R_4 (R_1 + R_2 + R_3) + R_3 (R_1 + R_2)}.$$

\[\text{\textsuperscript{8}}\text{A quick aside about signs. Consider this branch:}\]

Notice that we always label branch variable ($i$ and $v$) orientation so the direction of positive $i$ enters the the positive $v$ terminal. With the shown current arrow orientation, the current flowing into node $A$ is $-i$, while the current flowing into node $B$ is $i$. To find $i$, we then apply Ohm’s law $i = Gv$ where $v = e_A - e_B$ to write $i$, the current flowing into $B$, as $G(e_A - e_B)$. The current into $A$ is thus $-G(e_A - e_B).$
The question originally asked us to determine the current $i_3$, which, from Ohm’s law, is quick to find once we know $e_1$, $i_3 = -e_1/R_3$.

Substituting in the equation above, we find:

$$i_3 = \frac{(I_1R_1R_4 + V_2(R_1 + R_2) - R_4V_1)}{R_4(R_1 + R_2 - R_3) + R_3(R_1 + R_2)}.$$

**Conclusions**

There are a wide range of methods for simplifying circuits and intuiting circuit behaviors that will be introduced later on, and circuit elements with more complicated constitutive relations (like capacitors and inductors) will also be introduced. So when the node method is applied to such circuits are a lot harder to solve than the linear equations shown here, but the node method can still be applied to these complicated systems.

Really, the node method is the last analytic method you’ll need to learn to analyze the circuits in 6.002. In terms of pure analytic circuit tools, having mastered the node method, you should be able to at least construct equations (some of which may be hard to solve) for whatever comes along next.

Finally, it is worth mentioning that although we used the node method here as an exercise, to teach you the proper approach, these problems are more readily solved by using superposition. Indeed, by using superposition we could have avoided the heavy required algebra and written down the solutions more or less by inspection, with a minimum of analytic effort.

**Glossary and Definitions**

**Floating Voltage Source**  A voltage source where neither terminal is connected directly to ground. To be avoided when applying node method—if floating voltage source cannot be avoided, use supernodes to accommodate in node method.

**Ground**  Node chosen to have potential set to zero. Choice of ground in circuit analysis (as opposed to real circuits) is arbitrary, and can be selected so that analysis is simplified. For example, in the node method we typically select the node connected with shorts to the most voltage sources to be ground.

**Node Equations**  Linear equations that result from using the node method. If $N$ is the total number of nodes, and $M$ is the number of voltage sources, in general there will be $N - M$ such equations.
Node Method  Method of analyzing circuit using node potentials. Alternative to the loop method (see Agarwal & Lang if you’re interested in learning about the loop method).

Potential or Node Potential  Value of electrostatic potential at a node. Differentiated from voltage, which is a potential difference between two nodes.

Subcircuit  Collection of connected circuit components that can be enclosed in contiguous boundary within a larger circuit.

Standard Form  Circuit drawing style in which signals move from left to right (inputs on the left, outputs on the right), ground is placed at the bottom of the circuit, and the number of unconnected wire crossings are minimized. Sources are typically drawn vertical. Not all circuits will be presented in standard form, and in some cases redrawing in standard form greatly simplifies the acquisition of intuition about a circuit.

Supernode  An enclosed subcircuit used in Kirchhoff’s current law as if it were a node, i.e. the currents into or out of it are summed and set to zero.

Voltage  or Branch Voltage: Potential difference between two nodes in a circuit.