Problem 1

A. Low freq  Gain = 1/2, phase = 0
   Mid freq  phase = 90°
   High freq  Gain = 1, Phase = 90°

   NOTE  \( \frac{V_o}{V_i} = \frac{2R^2 + \omega^2 L^2 + j\omega L}{4R^2 + \omega^2 L^2} \)

B. Low freq  gain \( \propto \omega^2 \), phase = 180°
   Mid freq  phase = 0°
   High freq  gain = 1, phase 0°

   NOTE  \( \frac{V_o}{V_i} = \frac{-\omega^2 LC}{R - \omega^2 LC + j\omega L} \)

C. \( i = j\omega C V_i \rightarrow \) \( v_o = (-j\omega L)(j\omega C) V_i \)

   \( v_o = \omega^2 LC V_i \)  
   \( \Rightarrow \) gain \( \propto \omega^2 \)
   phase 0°

D. Bandpass: Low freq  gain \( \propto \omega \), phase = 90°
   Mid freq  gain = 1, phase 0°
   High freq  gain \( \propto \omega \), phase = -90°
Problem 2 Part (A)
Since the op-amp is ideal and there is negative feedback, we know that the voltages at the two input terminals of the op-amp are equal, and the voltage at the positive input terminal, \( v_+ \), is 0 since it is connected to ground. Therefore, \( v_- = v_+ = 0 \). Also, since the op-amp is ideal, there is no current flowing into or out of either of the op-amp’s input terminals. Therefore, \( i_- = i_+ = 0 \). Voltage drop across the resistor \( R \) is \( v_R = v_I - v_- = v_I - 0 = v_I \), and the current flowing through the resistor \( R \) is
\[
\frac{i_R}{R} = \frac{v_I}{R}
\]
Since \( i_- = 0 \), KCL at the op-amp’s negative input terminal (node) results in:
\[
i_R = i_- + i_B = 0 + i_B = i_B
\]
\[
\Rightarrow i_B = i_R = \frac{v_I}{R}
\]
Also, \( v_- - v_B = v_O \), and \( v_- = 0 \). Therefore, \( v_O = -v_B \).
From the i-v characteristic of the non-linear resistor we know
\[
i_B = k \cdot v_B^3
\]
\[
\Rightarrow v_B = \left( \frac{i_B}{k} \right)^{\frac{1}{3}}
\]
Since \( v_O = -v_B \), we get:
\[
v_O = -v_B = -\left( \frac{i_B}{k} \right)^{\frac{1}{3}}
\]
Substituting \( i_B = \frac{v_I}{R} \) into the equation above yields a relation between \( v_O \) and \( v_I \):
\[
\Rightarrow v_O = -\left( \frac{v_I}{R \cdot k} \right)^{\frac{1}{3}}
\]
Since \((-1)^{\frac{1}{3}} = -1\), \( v_O = (\frac{-v_I}{R \cdot k})^{\frac{1}{3}} \) is also an accepted solution.

Problem 2 Part (B)
Since the op-amp is ideal and there is negative feedback and \( v_+ = 0 \) as the positive input terminal of the op-amp is grounded, we get \( v_- = v_+ = 0 \). Also, since the op-amp is ideal, there is no current flowing into or out of either of the op-amp’s input terminals. Therefore, \( i_- = i_+ = 0 \). Voltage drop across the non-linear resistor is \( v_B = v_I - v_- = v_I - 0 = v_I \). Therefore, \( v_B = v_I \). KCL at the op-amp’s negative input terminal (node) results in:
\[
i_B = i_- + i_R = 0 + i_R = i_R
\]
\[
\Rightarrow i_R = i_B = k \cdot v_B^3
\]
Substituting \( v_B = v_I \) into the expression above for \( i_R \) yields:
\[
i_R = k \cdot v_I^3
\]
Also, \( v_- - i_R \cdot R = v_O \), and \( v_- = 0 \). Substituting \( i_R = k \cdot v_I^3 \) into this expression yields a relation between \( v_o \) and \( v_I \):
\[
\Rightarrow v_O = -R \cdot k \cdot v_I^3
\]
1
Problem 2 Part (C)
The frequency dependent impedance of the capacitor is $Z_c = \frac{1}{j\omega C}$. Since the op-amp is ideal and there is negative feedback and $v_+ = 0$ as the positive input terminal of the op-amp is grounded, we get $v_- = v_+ = 0$. Also, since the op-amp is ideal, there is no current flowing into or out of either of the op-amp’s input terminals. Therefore, $i_- = i_+ = 0$. Since $i_- = 0$, all of the current flowing through $R_1$ and $C$ flows through $R_2$. Therefore, using the node method at the $v_-$ terminal node of the op-amp results in:

$$\frac{v_i - 0}{R_1 + \frac{1}{j\omega C}} = \frac{0 - v_o}{R_2}$$

Rearranging the above equation yields

$$\frac{v_o}{v_i} = -\frac{R_2}{R_1 + \frac{1}{j\omega C}} = -\frac{j\omega CR_2}{j\omega CR_1 + 1}$$

Since we are concerned with sinusoidal drive, $v_i \equiv v_i(j\omega)$ and $v_o \equiv v_o(j\omega)$. Therefore, the expression above becomes:

$$\frac{v_o(j\omega)}{v_i(j\omega)} = -\frac{R_2}{R_1 + \frac{1}{j\omega C}} = -\frac{j\omega CR_2}{1 + j\omega CR_1}$$

Problem 2 Part (D)
Note that a steady-state output voltage is requested for this part and that there is both a DC and an AC component to the input voltage. Since the circuit is linear, we can use superposition to find the output voltage due to each component of the input voltage.

First, we can find the steady-state DC output voltage by analyzing the given circuit. At steady state for DC inputs, the capacitor acts like an open-circuit, resulting in no current flowing from the input through $R_1$ and $C$. As a result, no current flows through $R_2$ either and since the op-amp is ideal, $i_-$ is also zero. Therefore, $v_- = 0 \cdot R_2 = v_o,DC$ and since $v_- = v_+ = 0$, we get $v_o,DC = 0 [V]$.

To find the sinusoidal steady state response to $v_i(t) = 0.5sin(50000t)$ [V] we need to evaluate the magnitude and phase of the transfer function found in Part (c) at $\omega = 50000$ rad/sec, with $R_1 = 10$ k$\Omega$, $R_2 = 20$ k$\Omega$, and $C = 100$ nF:

$$\frac{v_o(j\omega)}{v_i(j\omega)} = -\frac{j\omega CR_2}{1 + j\omega CR_1}$$

The magnitude of the transfer function evaluates to:

$$\left|\frac{v_o(j\omega)}{v_i(j\omega)}\right| = \left|\frac{j\omega CR_2}{1 + j\omega CR_1}\right| = \frac{\omega CR_2}{\sqrt{1 + (\omega CR_1)^2}} = 1.9996$$

This means that the amplitude of the sinusoidal steady-state output is:

$$|v_o(j \cdot 50000)| = 1.9996 \cdot |v_i(j \cdot 50000)| = 1.9996 \times 0.5 = 0.9998 \approx 1$$

Note: We can check if the DC steady state solution we obtained above is correct by evaluating the transfer function magnitude at DC, i.e., $\omega = 0$:

$$v_o,DC = |v_o(j \cdot 0)| = \left|\frac{j\omega CR_2}{1 + j\omega CR_1}\right| \cdot |v_i(j \cdot 0)| = \frac{0 \times CR_2}{\sqrt{1 + (0 \times CR_1)^2}} \cdot 1 = 0$$
The phase of the transfer function evaluates to:

\[
\angle \frac{v_o(j\omega)}{v_i(j\omega)} = \angle \left( \frac{-j\omega CR_2}{1 + j\omega CR_1} \right) = \angle(-j\omega CR_2) - \angle(1 + j\omega CR_1) = -\frac{\pi}{2} - \tan^{-1}\left( \frac{\omega CR_1}{1} \right)
\]

\[
\Rightarrow \angle \frac{v_o(j\omega)}{v_i(j\omega)} = -\frac{\pi}{2} - 1.5507 = -3.121 \approx -\pi
\]

Therefore, the total steady state output can be found as the sum of the DC steady-state output and the sinusoidal steady state output:

\[
v_o(t) = 0 + 0.9998 \sin(50000t - 3.121) \approx \sin(50000t - \pi)
\]

\[
\Rightarrow v_o(t) \approx \sin(50000t - \pi) = -\sin(50000t) \text{ [V]}
\]
**Problem 3 (26 pts).**

In preparation for Festivus, George decided to string up some lights around (but not on) the aluminum Festivus pole. Unfortunately, he realized all he had available to power the lights was a Teensy board. Undaunted, George finds a capacitor and inductor in his apartment and creates the following circuit:

\[ v_s \] represents the Teensy output voltage and \( R \) represents the resistance of the lights. While the inductor is marked with \( L = 50 \text{ mH} \), the capacitor markings are too small to discern. Elaine comes in, removes the lights from the circuit (i.e., removes \( R \)), applies \( v_s = V_0 u(t) \) to the circuit, and measures the following response:

She tells George that he can figure out the capacitance from the response.

**A.** Determine the undamped resonant angular frequency \( \omega_0 \) for this circuit.

Period \( T' = 10\pi \text{ \mu s} = \pi 10^{-5} \text{s} \). Therefore, \( f = \frac{1}{T'} = \frac{10^5}{\pi} \text{ Hz} \). \( \omega_0 = 2\pi f = \frac{2\pi 10^5}{\pi} = 2 \cdot 10^5 \text{ rad/sec} \).

\[ \omega_0 = 2 \cdot 10^5 \text{ rad/sec} \]
B. Determine the capacitance $C$.

Since there is no resistance, we know that $\omega = \omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow C = \frac{1}{\omega_0^2L} = \frac{1}{(2 \cdot 10^5)^2 \cdot 50 \cdot 10^{-3}} = 5 \cdot 10^{-10} F = 500 \text{ pF} = 0.5 \text{ nF}.$

$C = 500 \text{ pF}$

George finds a capacitor on the floor with $C = 5 \text{ pF}$. He replaces the unknown capacitor with this new capacitor, and reattaches the lights to the circuit (i.e., reattaches $R$), creating the following circuit:

He applies a 3.3 Vpp sinusoid at a frequency of $1/\pi$ MHz, and measures the following response:

Elaine takes a look and says, “nice Q”.
C. Determine the transfer function of this circuit **analytically**, i.e., determine $H(j\omega) = v_R(j\omega)/v_s(j\omega)$ in terms of $R$, $L$, and $C$ (do not substitute in values).

Note that this is the same circuit used in Lab 6. This circuit was also analyzed in lecture on Tue Nov 14,

\[
H(s) = \frac{R \parallel \frac{1}{sC}}{R \parallel \frac{1}{sC} + sL} = \frac{R \left(\frac{1}{sC}\right)}{R + \frac{1}{sC}} = \frac{R}{sRC + 1} = \frac{R}{R + sL(sRC + 1)}
\]

\[
H(s) = \frac{R}{s^2RLC + sL + R} = \frac{1}{LC} \frac{1}{s^2 + s \frac{1}{RC} + \frac{1}{LC}}
\]

\[
H(j\omega) = \frac{1}{LC} \frac{1}{-\omega^2 + j\omega \frac{1}{RC} + \frac{1}{LC}}
\]

D. Determine analytically $H(j\omega_0)$.

When $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$, then we see that the $-\omega^2$ term in the denominator cancels the $\frac{1}{LC}$ term, leaving:

\[
H(j\omega_0) = \frac{1}{\sqrt{LC}} \frac{1}{1} = -\frac{jR}{\sqrt{LC}} = -jR C \omega_0 = -\frac{jR}{\sqrt{C}} = -\frac{jR}{Z_0}
\]

\[
H(j\omega_0) = -\frac{jR}{Z_0}
\]
E. Determine numerically the quality factor $Q$.

We did this in Lab 6, where we realized that when driven at resonance, the voltage gain was equal to $Q$, and we measured the voltage gain in our circuit to determine the $Q$ of our circuit in the lab, and then to determine $R$ (as in part F).

One way to see this is to put the TF in standard form:

$$H(j\omega) = \frac{1}{LC} - \frac{1}{-j\omega \frac{1}{RC} + \frac{1}{LC} + \frac{\omega_0^2}{Q}} = \frac{\omega_0^2}{-\omega^2 + j\omega + \frac{\omega_0^2}{Q} + \frac{1}{LC}}$$

When put in this form, we see that when $\omega = \omega_0$, the TF reduces to $H(j\omega_0) = \frac{\omega_0^2}{j\omega_0\frac{\omega_0}{Q}} = -jQ$

We also know that $H(j\omega_0) = v_R(j\omega)/v_s(j\omega)$. So if we find the voltage gain at the resonant frequency, that is directly $Q$. In this case, the input voltage is 3.3 V$_{pp}$ and the output voltage is 2*19.8 V$_{pp}$, therefore $Q=12$.

Note that it is not correct to say that because the signal goes on forever that the $Q$ is infinite or that there is no damping. There is clearly a resistor in the circuit, and so there is damping. Remember that this is a sinusoidal steady-state problem, not a transient response. In other words, we are continually putting in energy. If this were a transient response and we observed this waveform, then the $Q$ would indeed be infinite and there would be no damping. But since we’re driving with a sine wave, as long as the system is LTI we will get a sine wave out, even if there are only resistors in the circuit; if this was a voltage source connected to a resistor, the output would be a sine wave.

$$Q = 12$$

F. Determine numerically the light resistance $R$.

We did this in lab as well. If we know $Q$, we can get back out $R$, because we see from above that $Q = \frac{R}{Z_0} = \frac{R}{\sqrt{LC}}$

Since we know $Q$, $L$, and $C$, we can determine $R$. Note that some students used $Q = \frac{Z_0}{R}$. This equation is not correct. The specific relationship between $Q$ and the circuit parameters depends on the topology of the particular circuit.

$$R = 1.2 \text{ M}\Omega$$
G. George wants the lights to be brighter and so the voltage to be higher. He asks if replacing his current inductor with a new inductor of \( L = 20 \) mH will increase or decrease the light brightness, assuming there are no limitations on the current sourced by the Teensy and that the drive frequency can be altered to the new resonant frequency.

Since \( Q = \frac{R}{Z_0} = \frac{R}{\sqrt{LC}} \), decreasing \( L \) will decrease \( Z_0 \), which will increase \( Q \). Since \( Q \) gives the voltage gain at resonance. This will increase brightness. In Lab 6 we were trying to design \( L \) and \( C \) to maximize voltage gain and so be able to light up the longest string of lights.

Circle one:  
Changing to \( L = 20 \) mH will **decrease** light brightness

Changing to \( L = 20 \) mH will **increase** light brightness

1-2 sentence explanation:

*See above.*
Problem 4 (26 pts).

Consider the following parallel RLC circuit, with current input \( i(t) \).

\[
\begin{align*}
I & \quad R \quad L \quad C \\
\quad & \quad + \quad v_{out} \\
\quad & \quad -
\end{align*}
\]

A. Determine the transfer function \( v_{out}(j\omega)/I(j\omega) \).

\[
\frac{v_{out}}{I} = \frac{\frac{R\cdot s\cdot L}{R+sL} \cdot \frac{1}{sC}}{\frac{1}{sC} \cdot \frac{LR\cdot s}{LRs + R+sL} \cdot \frac{1}{sC} = \frac{LRs}{RLCs^2 + R + sL} = \frac{Ls}{LCs^2 + \frac{L}{R} + s + 1}}
\]

\[
\frac{v_{out}(j\omega)}{I(j\omega)} = \frac{j\omega L}{1 - \omega^2 LC + j\omega \frac{L}{R}}
\]

\[
v_{out}(j\omega)/I(j\omega) = \frac{j\omega L}{1 - \omega^2 LC + j\omega \frac{L}{R}}
\]

B. Does this circuit behave as a low-pass, high-pass, band-pass, or band-stop (notch) filter?

Circle one: Low-pass  High-pass  Band-pass  Band-stop
Problem 4 (26 pts).

Consider the following parallel RLC circuit, with current input $I(t)$.

A. Determine the transfer function $v_{out}(j\omega)/I(j\omega)$.

$$
v_{out} = G_{tot} \cdot I
$$

$$
G_{tot} = \frac{1}{R} + \frac{1}{j\omega L} + \frac{1}{j\omega C}
$$

$$
\therefore \frac{v_{out}(j\omega)}{I(j\omega)} = \frac{1}{\frac{1}{R} + j\omega C + \frac{1}{j\omega L}} = \frac{j\omega L}{1 - \omega^2LC + j\omega \frac{1}{R}}.
$$

B. Does this circuit behave as a low-pass, high-pass, band-pass, or band-stop (notch) filter?

$$
\omega \to 0 \quad \frac{V_{out}}{I} \to 0 \quad \omega(0)
$$

$$
\omega \to \infty \quad \frac{V_{out}}{I} \to \frac{1}{\omega} \quad (0)
$$

Circle one: Low-pass High-pass Band-pass Band-stop
For the remainder of this problem, assume $L = 8 \text{ mH}$, $C = 500 \text{ pF}$, and $R = 20 \text{ kΩ}$.

C. Sketch the magnitude and phase of $\frac{v_{\text{out}}(j\omega)}{I(j\omega)}$ on the axes below. Label important frequencies, values, and slopes. The magnitude should be plotted on log-log axes, while the phase should be log-linear.

From the transfer function,

$$w_0 = \frac{1}{\sqrt{LC}} = \frac{500 \text{ kΩ}}{\sqrt{8 \times 500 \text{ pF}}} \approx \frac{50 \text{ kHz}}{2\pi} = 79.6 \text{ kHz}.$$  

$$Q = \sqrt{\frac{C}{L}} \cdot R = 5$$

$$\Delta \omega = \frac{w_0}{Q} = \frac{50 \text{ kHz}}{5} = 10 \text{ kHz}.$$  

High $Q$ 2nd order filter.

$\log(\omega_0) \approx 4 \log Q^2$.  

$\log(200 \text{ kHz})$.  

$\Delta \omega = 100 \text{ kHz}$.  

$5 \times 10^5$ or $5 \log 5$.  

Low freq: $\frac{V_o}{I} \approx jwL$.  

High freq: $\frac{V_o}{I} \approx -\frac{j}{\omega C} = \frac{1}{jwC}$ (90°).

$90^\circ$.  

$-90^\circ$.  

The diagram shows the magnitude and phase responses as functions of frequency on log-log axes.
D. Assume now that the input is \( I(t) = I_0 u(t) \) where \( I_0 = 1 \text{ mA} \). Sketch \( v_{out}(t) \) on the axes below. Label important times and values, including \( v_{out}(0), \frac{dv_{out}}{dt} \bigg|_{t=0}, \) and \( v_{out}(\infty) \).

\[
Q = 5 \Rightarrow \frac{1}{2} \text{ underdamped, damping within } n \text{ periods.}
\]

Also \( \omega_d = \omega_0 \) (\( \omega_0 \rightarrow \omega \)).

\( v_{out}(0) = 0 \) (high freq. component not passing)

\( v_{out}(\infty) = 0 \) (low freq. (DC) component not passing)

At \( t=0^+ \), all the current passes through \( C \)

\[
\frac{I_0}{C} = \left. \frac{dv_0}{dt} \right|_{t=0} = \frac{1 \text{ mA}}{500 \times 10^{-12} \text{ F}} = 2 V/\text{ms}.
\]