RLC Filters

![RLC Filters Diagram]

TF 1

\[
\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{j\omega L}{R + \frac{j\omega L}{1 - \omega^2 LC}} = \frac{j\omega L}{R - \omega^2 R L C + j\omega L} = \frac{j\omega}{R C + j\omega - \omega^2}
\]
TF 2

\[
\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{R}{R + \frac{j\omega L}{1 - \omega^2 LC}} = \frac{R - \omega^2 RLC}{R - \omega^2 RLC + j\omega L} = \frac{1}{\frac{1}{LC} - \frac{\omega^2}{RC - \omega^2}}
\]

TF 3

\[
\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{j\omega RC}{j\omega RC - \omega^2 LC + 1} = \frac{j\omega R}{\frac{1}{LC} + \frac{j\omega R}{L} - \omega^2}
\]
The natural resonant frequency of the system is given by:

$$f(\omega) = \frac{1}{\omega_0^2 + j\omega_0 \alpha}$$

where:

- $\omega_0$: Natural Resonant Frequency of System
- $\alpha$: Damping Coefficient

The relationship between $\alpha$ and $\omega_0$ dictates how the circuit will respond in time (but also frequency)!
RLC Circuits (Examples of Second Order)

- Undriven (series) RLC circuit
  
  \[ \frac{d^2v(t)}{dt^2} + \frac{R}{L} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = 0, \]

  Solution has form: \[ A e^{st}, \]

  Characteristic equation: \[ s^2 + \frac{R}{L} s + \frac{1}{LC} = 0, \]

  Damping factor: \[ \alpha = \frac{R}{2L}, \]

  Undamped resonant frequency: \[ \omega_0 = \frac{1}{\sqrt{LC}}. \]

  For series RLC Depends on topology!!

  Characteristic impedance \[ Z_0 = \frac{V_{cph}}{I_{dph}}, \]

  Homogeneous response \[ v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}. \]

Second Order Response

- Three cases dictated by \( \omega_0 \) and \( \alpha \)

  - \( \alpha < \omega_0 \) Underdamped
    
    \[ s_1 = -\alpha + j\omega_0, \quad s_2 = -\alpha - j\omega_0, \]
    
    \[ v_{CH} = e^{-\alpha t} (A e^{j\omega_0 t} + A_0 e^{-j\omega_0 t}), \]
    
    \[ v_{CH} = e^{-\alpha t} (A \cos(\omega_0 t) + A_0 \sin(\omega_0 t)), \]
    
    \[ v_{CH} = A e^{-\alpha t} \cos(\omega_0 t) + A_0. \]

  - \( \alpha = \omega_0 \) Critically damped
    
    \[ s_1 = s_2 = -\alpha, \]
    
    \[ v_{CH} = A e^{-\alpha t}. \]

  - \( \alpha > \omega_0 \) Overdamped
    
    \[ s_1 = -\alpha + j\omega, \quad s_2 = -\alpha - j\omega, \]
    
    \[ v_{CH} = e^{-\alpha t} A_1 e^{\omega t} + A_2 e^{-\omega t}. \]

  Quality factor \[ Q = \frac{\omega_0}{2\alpha}. \]
RLC circuit

\[ \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{\frac{j \omega R}{L}}{1 + \frac{j \omega R}{L} - \omega^2} \]

\[ \omega = 0 \quad \omega = \omega_0 \quad \omega = \infty \]

- \[ 0 \angle 0 \quad 1 \angle 0 \quad 0 \angle \pi / 2 \]

- \[ 0 \angle 0 \quad 0 \angle +\pi / 2 \quad 1 \angle 0 \]

- \[ 0 \angle \pi / 2 \quad 1 \angle 0 \quad 0 \angle \pi / 2 \]

- \[ 1 \angle 0 \quad 0 \angle +\pi / 2 \quad 1 \angle 0 \]

- \[ \text{• Can tune sharpness of resonance by changing } R \]
  - Small R \( \rightarrow \) blunt resonance
  - Large R \( \rightarrow \) sharp resonance
Band-pass filters
• Combine R and parallel LC or series LC

Bandwidth
\[ BW = \frac{\omega_0}{Q} = 2\alpha \]

Band-stop or Notch filters
• Combine R and parallel LC or series LC
Sallen Key Topology

https://en.wikipedia.org/wiki/Sallen%E2%80%93Key_topology
KCL at $e_-$:
\[
\frac{v_I(t)}{R} - \frac{v_{out}(t)}{C} = \frac{d}{dt}(0 - v_{out}(t))
\]
\[
\int_{t_1}^{t_2} \frac{v_I(t)}{R} dt = \int_{t_1}^{t_2} C \frac{d}{dt} v_{out}(t) dt
\]
\[
-\frac{1}{RC} \int_{t_1}^{t_2} v_I(t) dt = -\frac{1}{RC} \int_{t_1}^{t_2} v_{out}(t) dt = -v_{out}(t)
\]
\[
v_{out}(t) = -\frac{1}{RC} \int_{t_1}^{t_2} v_I(t) dt
\]

Also analyze in freq domain:
\[
\frac{v_o}{v_i} = -\frac{Z_2}{Z_1}
\]
If $Z_1 = R$, $Z_2 = \frac{1}{j\omega C}$
\[
\frac{v_o}{v_i} = -\frac{1}{j\omega C}
\]
\[
\frac{1}{j\omega RC}
\]
KCL at $e_1$:

$$\frac{e_1 - v_{in}}{Z_1} + \frac{e_1 - v_{out}}{Z_2} + \frac{e_1 - e_2}{Z_3} = 0 \quad \text{(come back to it)}$$

KCL at $e_2$:

$$\frac{e_2 - e_1}{Z_2} + \frac{e_2}{Z_4} = 0 \quad \Rightarrow \quad e_2 \left( \frac{1}{Z_2} + \frac{1}{Z_4} \right) = \frac{e_1}{Z_2} \quad e_2 \frac{Z_2 + Z_4}{Z_4 Z_2} = \frac{e_1}{Z_2} \quad e_1 = \frac{Z_2 + Z_4}{Z_4} e_2$$

Note about op amp in negative feedback (path from $v_{out}$ to inverting input)... Hence $v_- = v_+ = v_{out}$

therefore $e_0 = v_{out}$ Henceforth $e_1 = \left( \frac{Z_2 + Z_4}{Z_4} \right) v_{out}$

Now return to our first KCL eq:

$$\begin{align*}
v_{out} \left( \frac{Z_2 + Z_4}{Z_1 Z_4} \right) - \frac{v_{in}}{Z_1} + v_{out} \left( \frac{Z_2 + Z_4}{Z_2 Z_3} \right) - \frac{v_{out}}{Z_2} + v_{out} \left( \frac{Z_2 + Z_4}{Z_3 Z_4} \right) - \frac{v_{out}}{Z_4} &= 0 \\
v_{out} \left( \frac{Z_2 + Z_4}{Z_2 a} \right) \left( \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right) - \frac{1}{Z_2} - \frac{1}{Z_3} &= \frac{v_{in}}{Z_1} \\
v_{out} \left( \frac{Z_2 + Z_4}{Z_4} \right) \left( \frac{Z_2 Z_3 + Z_2 Z_4 + Z_3 Z_4}{Z_1 Z_2 Z_3} \right) - \frac{Z_2 Z_4}{Z_1 Z_2 Z_3 Z_4} - \frac{Z_2 Z_3}{Z_1 Z_2 Z_3 Z_4} &= \frac{v_{in}}{Z_1} \\
v_{out} \left( \frac{Z_2 Z_3 + Z_2 Z_4 + Z_3 Z_4}{Z_3 Z_4} \right) - \frac{Z_2 Z_4}{Z_1 Z_2 Z_3 Z_4} - \frac{Z_2 Z_3}{Z_1 Z_2 Z_3 Z_4} &= -v_{in} \\
v_{out} \left( \frac{Z_2 Z_3 + Z_2 Z_4 + Z_3 Z_4}{Z_4} \right) &= \frac{v_{in}}{Z_1 Z_2 Z_3 Z_4} \\
v_{out} = \frac{v_{out}}{v_{in}} &= \frac{Z_2 Z_4}{Z_1 Z_2 Z_3 + Z_2 Z_4 + Z_2 Z_3 Z_4 + Z_3 Z_4 Z_4}
What you drop in for each impedance block will affect the output. Certain patterns result in certain types of filters!

\[ \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R_1R_2}{(\frac{1}{j\omega C_1}) + (\frac{1}{j\omega C_2})(R_1) + (\frac{1}{j\omega C_2})(R_2) + R_1R_2} \]

\[ \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{\frac{-\omega^2 C_1 C_2 R_1 R_2}{1 + j\omega C_2 R_1 + j\omega C_1 R_2 - \omega^2 C_1 C_2 R_1 R_2}}{1 - \frac{\omega^2}{C_1 C_2 R_1 R_2}} \]

Is this high pass? Let's do a quick check.

(\text{\textcircled{1}}) \quad \omega \rightarrow 0 \quad \frac{V_{\text{out}}}{V_{\text{in}}} \rightarrow \quad \text{magnitude: } 0 \quad \text{phase: } \frac{\pi}{2} \quad \text{Low at Low } \omega

(\text{\textcircled{2}}) \quad \omega \rightarrow \omega_0 \quad \frac{V_{\text{out}}}{V_{\text{in}}} \rightarrow \quad \text{Value at } \omega_0 \text{...not a peak in highpass.}

(\text{\textcircled{3}}) \quad \omega \rightarrow \infty \quad \frac{V_{\text{out}}}{V_{\text{in}}} \rightarrow \quad \text{magnitude: } 1 \quad \text{phase: } \angle 0 \quad \text{high at high } \omega
Frequency Response:

\[ \frac{\omega}{\omega_0} \]  

\[ 20 \log \left( \frac{\omega}{\omega_0} \right) \]  

\[ \log(\omega) \]

\[ w_c = \omega_0 \]

\[ \omega_0 > \alpha \]

Little damping

\[ +40 \text{dB/decade} \]

\[ w_p \approx w_c \text{ which is } < \omega_0 \]

\[ \omega_d = \sqrt{\omega_0^2 - \alpha^2} \]

\[ \omega_c = \omega_0 \]

\[ \alpha = \omega_0 \]

Critical damping

\[ +40 \text{dB/decade} \]

\[ \text{peak occurs at damped res freq } \omega_d \]

\[ w_0 \neq w_c \]

\[ \text{peak occurs very close to resonant } \omega_0 \]

Down 6dB as opposed to 3dB at critical damping because this is a second order filter!
Changing op amp feedback path from a regular old buffer (gain of 1) to something with some resistors in the feedback path gives us freedom to increase the overall GAIN of the system $R_3/R_4$ with the op amp form a non-inverting amplifier, and if you were to drop the implication of that into our original derivation, you'd find that the original transfer function just gets scaled by this gain. In terms of frequency response if you have a high-pass filter

$$\frac{v_{out}}{v_{in}} = \left(1 + \frac{R_3}{R_4}\right)H(j\omega)$$

And the Sallen Key is super flexible! Change what parts you put where and you'll get a Low Pass Filter!

EX09